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Approximating Laplace transforms

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1 Introduction

In mathematics, the Laplace transform is a powerful technique which helps to solve complex differential equations by converting those into simpler algebraic equations. In this paper, we study the Laplace transform, properties of transforms, and how they can be approximated for functions with no or complex Laplace transform.

2 The Laplace transform

The Laplace transform turns a function into another function by a given rule.

Definition 1 (Laplace Transform [5])**.** Let f be a function defined for $t \geq 0$.
The *Laplace transform* of f is defined by the improper integral

$$\mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

provided that the integral converges.

The integral above is calculated by the usual method of improper integrals:

$$\mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)\, dt\,.$$

The Laplace transforms of some functions can be calculated from this definition.

Example. For $s \in (0, \infty)$,

$$\mathscr{L}\{1\}(s) = \lim_{b\to\infty} \int_0^b e^{-st} \cdot 1\, dt = \lim_{b\to\infty} -\frac{e^{-st}}{s}\bigg|_0^b = \lim_{b\to\infty} \left(-\frac{e^{-sb}}{s} + \frac{1}{s} \right) = \frac{1}{s}\,.$$

Example. For $s \in (1, \infty)$,

$$\begin{aligned}\mathscr{L}\{e^t\}(s) = \lim_{b\to\infty} \int_0^b e^{-st} e^t\, dt = \lim_{b\to\infty} \int_0^b e^{(1-s)t}\, dt &= \lim_{b\to\infty} \frac{e^{(1-s)t}}{1-s}\bigg|_0^b \\ &= \lim_{b\to\infty} \left(\frac{e^{(1-s)b}}{1-s} - \frac{1}{1-s} \right) = \frac{1}{s-1}\,.\end{aligned}$$

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Example 2. *The Laplace transforms of some basic functions are as follows, for suitable values of s:*

$$\begin{aligned}
\mathscr{L}\{1\}(s) &= \frac{1}{s} \\
\mathscr{L}\{t^n\}(s) &= \frac{n!}{s^{n+1}} \\
\mathscr{L}\{e^{at}\}(s) &= \frac{1}{s-a} \\
\mathscr{L}\{\sin(kt)\}(s) &= \frac{k}{s^2+k^2} \\
\mathscr{L}\{\cos(kt)\}(s) &= \frac{s}{s^2+k^2} \,.
\end{aligned}$$

The inverse transform

As the Laplace transform exists, its inverse transform also exists. A specific formula for the inverse transform does not exist; it can only be derived from the Laplace transform.

Definition 3 (Inverse Laplace transform [5]). If $F(s) = \mathscr{L}\{f\}(s)$, then

$$f(t) = \mathscr{L}^{-1}\{F(s)\}(t) \,.$$

is the *inverse Laplace transform* of $F(s)$.

Example. By Example 2, $\mathscr{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$, so

$$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}(t) = t^n \,.$$

Existence of the Laplace transform

Of course, the improper integral $\int_0^\infty e^{-st} f(t)\, dt$ might not exist. Then, when does the Laplace transform exist? We will propose a theorem giving an existence condition. First, we first define *exponential order*.

Definition 4 (Exponential order). A function f is of *exponential order* when constants a, $k > 0$ and $T > 0$ exist such that

$$f(t) \le ke^{at} \quad \text{when} \quad t > T \,.$$

This means that f should be eventually smaller than an exponential function. For example, $f(t) = t^n$ is of exponential order for any natural number n, but $f(t) = e^{t^2}$ is not of exponential order.

The following theorem provides a sufficient condition for the existence of a Laplace transform.

Theorem 5. *Suppose that f is piecewise continuous on $[0, \infty)$ and of exponential order. Then the Laplace transform of f exists for $s > 0$.*

Proof. We divide the interval $[0, \infty)$ into the sub-intervals $[0, T)$ and $[T, \infty)$:

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt.$$

We see that $\int_0^T e^{-st} f(t)\, dt$ is finite. Since f is of exponential order, constants $a, k, T > 0$ exist so that

$$|f(t)| \leq ke^{at}\,.$$

for all $t > T$. Therefore,

$$\left| \int_T^\infty e^{-st} f(t)\, dt \right| \leq \int_T^\infty |e^{-st} f(t)|\, dt \leq k \int_0^\infty e^{-st} \cdot e^{at}\, dt = k\, \frac{e^{-(s-a)T}}{s-a} < \infty\,,$$

for all $s > a$. □

Some properties of the Laplace transform

Here are some properties that help evaluate Laplace transform of functions. [5]

Theorem 6 (Linearity of the Laplace transform)**.**
If $\mathscr{L}\{f_1\}$ and $\mathscr{L}\{f_2\}$ exist for $s > a_1$ and $s > a_2$. Then, for $s > max\{a_1, a_2\}$,

$$\mathscr{L}\{c_1 f_1 + c_2 f_2\}(s) = c_1 \mathscr{L}\{f_1(t)\}(s) + c_2 \mathscr{L}\{f_2(t)\}(s)\,.$$

Theorem 7 (Linearity of the inverse transform)**.** *The inverse Laplace transform is a linear transform. That is, for constants c_1 and c_2,*

$$\mathscr{L}^{-1}\{c_1 F(s) + c_2 G(s)\}(t) = c_1 \mathscr{L}^{-1}\{F(s)\}(t) + c_2 \mathscr{L}^{-1}\{G(s)\}(t)\,.$$

Theorem 8 (Transform of derivatives)**.** *If f' is continuous on $[0, \infty)$ and f is of exponential order, then*

$$\mathscr{L}\{f'(t)\}(s) = sF(s) - f(0)\,.$$

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order, and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)\,.$$

3 Taylor series

Using the Taylor series is our main method of approximating Laplace transforms. To know what the Taylor series is, one should start with power series.

Definition 9 (Power series [3])**.** A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

where a_n represents the coefficient of the nth term and c is a constant.

Taylor series is a special case of power series, where the coefficients are determined by a formula.

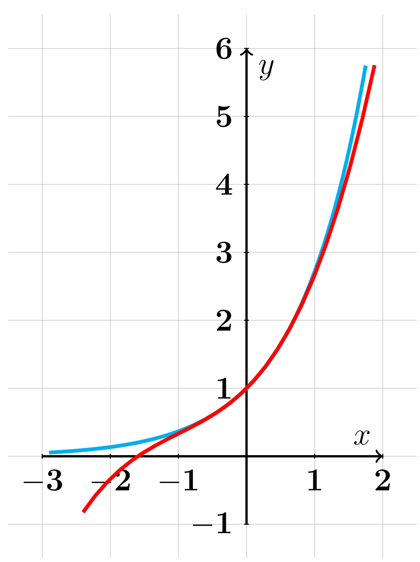
Definition 10 (Taylor series [4])**.** A *Taylor series* of a function f that is infinitely differentiable at a real number a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

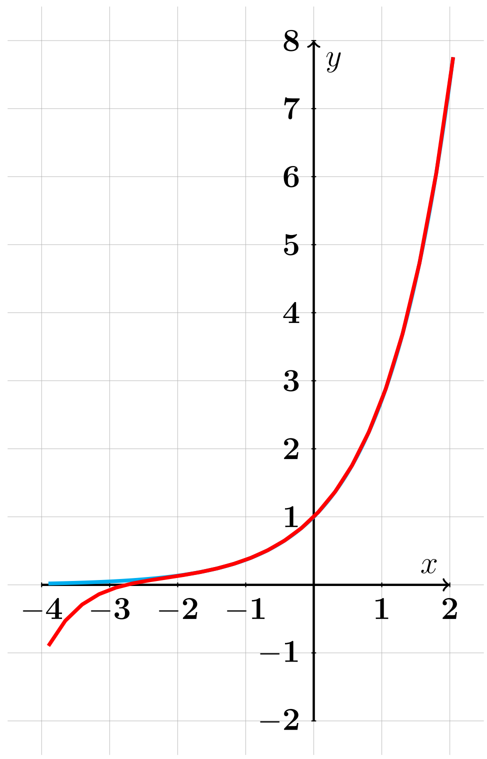
where $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a. This polynomial is called the *Taylor polynomial* of f.

Taylor's Theorem

One might think: what is the significance of the Taylor series? Its importance arises from the fact that it provides a good approximation of the original function. For example in the figure below, the (red) graph of $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ looks almost the same as the (blue) graph of $y = e^x$, at least when x is close to 0:



The figure above shows the graph of $y = e^x$ and its Taylor approximation. The blue curve represents $y = e^x$, and the red curve represents $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, which is a Taylor polynomial of e^x of degree 3. Notice that the Taylor polynomial of degree 3 looks nearly the same with the graph of $y = e^x$ near 0. Increasing the degree of the Taylor polynomial will increase its accuracy, providing a better approximation for the original function. The figure below shows the Taylor approximation of e^x with Taylor polynomial of degree 7, which is much more accurate than the Taylor polynomial of degree 3.



Summing the Taylor series to infinity, some Taylor series become the function itself that the Taylor series is derived from. One such example is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

for all real x. However, not all functions are equal to their Taylor series. The condition for a function to be equal to its Taylor series is stated in the *Taylor's theorem*.

Theorem 11 (Taylor's Theorem [2]). *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and*

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In the theorem above, $R_n(x)$ is the error, or remainder, between $f(x)$ and $T_n(x)$, and R is the radius of convergence. The theorem states that in its interval of convergence, if the error bound keeps decreasing so that the limit of it is 0, then the function is equal to its Taylor series.

The following theorem gives an upper bound on the size of the remainder $R_n(x)$.

Theorem 12 (Taylor's Inequality [2]). *If $|f^{(n+1)}(x)| \leq M$ whenever $|x-a| \leq d$, then*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

whenever $|x-a| \leq d$.

Example. Prove that e^x is equal to its Taylor series at 0.

Since $(e^x)^{(k)}(0) = 1$ for all k, the Taylor series of e^x at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

If d is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$. Letting $M = e^d$ therefore gives the inequality

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1}$$

for all x satisfying $|x| \leq d$. Using the lemma [2] that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

for every real number x, we get

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 .$$

Thus, $\lim\limits_{n\to\infty} R_n(x) = 0$ by the Squeeze Theorem, and so e^x is equal to its Taylor series.

Instead of using improper integrals directly, Laplace transforms of some functions can be calculated from the simple Laplace transform $\mathscr{L}\{t^n\} = n!/s^{n+1}$, using Taylor series:

Example. Since the Taylor series of e^t is $e^t = 1 + t + \frac{t^2}{2!} + \cdots$,

$$\mathscr{L}\{e^t\}(s) = \mathscr{L}\left\{1 + t + \frac{t^2}{2!} + \cdots\right\}(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{s^n} = \frac{\frac{1}{s}}{1 - \frac{1}{s}} = \frac{1}{s-1} .$$

Example. Since the Taylor series of $\sin t$ is $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$,

$$\mathscr{L}\{\sin t\}(s) = \mathscr{L}\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right\}(s)$$

$$= \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{-s^2}\right)^n = \frac{\frac{-1}{s^2}}{1 - \frac{-1}{s^2}} = \frac{1}{s^2+1} .$$

Approximating functions with its Taylor series

Instead of summing the Taylor series up to infinity, summing only up to a finite degree gives an approximation of the function which its Taylor series is derived from.

Definition 13 (Partial sum of Taylor series)**.**
Define $T_k(t)$ to be the kth partial sum of the Taylor series of $f(t)$. That is,

$$T_k(t) = \sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!} = f(a) + f'(a)(t-a) + \frac{f''(a)}{2!}(t-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(t-a)^k \,.$$

When $R_n(t) \to 0$ as $n \to \infty$, $T_n(t) \to f(x)$ as $n \to \infty$ because $f(t)$ is equal to its Taylor series. By Taylor's inequality, the remainder term $R_n(t)$ will get smaller as n increases, and $R_n(t) \approx 0$ near $t = a$ when n is sufficiently large. Therefore, we can approximate $f(t)$ by partial sums of Taylor series, and $f(t) \approx T_n(t)$ near $t = a$ when n is sufficiently large.

4 Approximating Laplace transforms

Section 2 stated that Laplace transform is defined by an improper integral. Then, does the Laplace transform exist for every function? The answer is no, since when the integral diverges, the Laplace transform does not exist. The Laplace transform of $1/t$ is one such example. Also, there are functions that, even if they are of exponential order, their Laplace transform cannot be expressed in terms of standard functions. For such cases, Laplace transforms can still be approximated. The idea is to express the function as its Taylor polynomial, and then apply the Laplace transform.

Theorem 14 (Laplace Remainder Theorem)**.** *Let $f_0, f_1, f_2, \ldots$ be a sequence of functions. If f_n is of exponential order for all n and if $\lim_{n\to\infty} f_n = 0$, then*

$$\lim_{n\to\infty} \mathscr{L}\{f_n\}(s) = 0 \,.$$

Proof. Since f_n is of exponential order for all n, there are constants $a, k, T > 0$ so that

$$f_n(t) \leq ke^{at}$$

when $t > T$. Then we have $e^{-st} f_n(t) \leq e^{-st} \cdot ke^{at}$. Since $\int_0^\infty e^{-st} \left| ke^{at} \right| dt < \infty$, the Dominated Convergence Theorem [1] implies that

$$\lim_{n\to\infty} \int_0^\infty f_n(t)\, dt = \int_0^\infty \lim_{n\to\infty} f_n(t)\, dt \,.$$

Therefore,

$$\lim_{n\to\infty} \mathscr{L}\{f_n\}(s) = \lim_{n\to\infty} \int_0^\infty f_n(t)\, dt = \int_0^\infty \lim_{n\to\infty} f_n(t)\, dt = \mathscr{L}\{0\}(s) = 0$$

which completes the proof. □

By Theorem 14, even though the Laplace transform shifts a function in t-domain to a function in s-domain, the limit of the sequence of the Laplace transform is 0, so the approximation is still valid in the s-domain.

Theorem 15 (Approximating Laplace transforms)**.** *If a function f is infinitely differentiable at 0, then its Laplace transform can be approximated by*

$$\mathscr{L}\{f\}(s) \approx \sum_{n=0}^{k} \frac{f^{(n)}(0)}{s^{n+1}} = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \cdots + \frac{f^{(k)}(0)}{s^{k+1}}$$

when t is in the interval of convergence and k is finite.

Proof. Since f_n is of exponential order because it is a polynomial, $R_n = f - T_n$ is also of exponential order. Since $\lim_{n\to\infty} R_n = 0$, Theorem 14 implies that

$$\lim_{n\to\infty} \mathscr{L}\{R_n\}(s) = 0\,.$$

Therefore, we have

$$\mathscr{L}\{f\}(s) \approx \mathscr{L}\left\{\sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n\right\}(s) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} \frac{n!}{s^{n+1}} = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{s^{n+1}}\,. \qquad \square$$

Even if $f(t)$ doesn't have a Laplace transform, then one can approximate a Laplace transform for $f(t)$ by the theorem above because the Taylor polynomial of finite degree derived from $f(t)$ always has a Laplace transform. Also, since the Taylor polynomial of $f(t)$ behaves like $f(t)$ near $t = 0$, we can approximate solutions of differential equations with initial values at 0 using Laplace transforms. One example is stated to clarify the usage of the theorem.

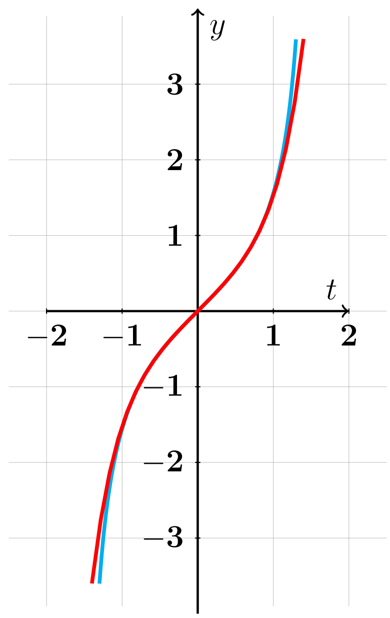
Example. Even though the Laplace transform of $\tan t$ cannot be expressed in terms of standard functions, the theorem above can be used to approximate the Laplace transform of $\tan t$ in its interval of convergence, which is $(-\pi/2, \pi/2)$. The derivation of a Taylor polynomial of degree 7 from $\tan t$ is as follows. Let $f(t) = \tan t$. Since

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= 2 \\ f^{(4)}(0) &= 0 \\ f^{(5)}(0) &= 16 \\ f^{(6)}(0) &= 0 \\ f^{(7)}(0) &= 272\,, \end{aligned}$$

the Taylor polynomial of $\tan t$ of degree 7 is

$$\tan t \approx t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7\,,$$

which is drawn in the figure below.



The red curve is the graph of $y = \tan t$ and the blue curve is the graph of the Taylor polynomial of $\tan t$ of degree 7. Note that the Taylor polynomial behaves exactly like $\tan t$ near $t = 0$.

Applying Laplace transforms on both sides gives

$$\mathscr{L}\{\tan t\}(s) \approx \mathscr{L}\left\{t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7\right\}(s) = \frac{1}{s^2} + \frac{2}{s^4} + \frac{16}{s^6} + \frac{272}{s^8}\,.$$

Therefore, the Laplace transform of $\tan t$ can be approximated as

$$\mathscr{L}\{\tan t\}(s) \approx \frac{1}{s^2} + \frac{2}{s^4} + \frac{16}{s^6} + \frac{272}{s^8}\,. \tag{1}$$

5 Solving differential equations

This approximation method can also be used to solve differential equations, especially for approximating solutions near $t = 0$ in initial-value problems. Suppose there is an initial-value problem involving $f(t)$. Instead of solving the initial-value problem with $f(t)$ included, solve with its Taylor polynomial; this will be much easier because the Laplace transform of the Taylor polynomial is simple. Also, since the Taylor polynomial of $f(t)$ behaves like $f(t)$ near $t = 0$, the solution of the initial-value problem evaluated from the Taylor polynomial of $f(t)$ will behave like it evaluated from $f(t)$.

Example. Solve $y'' + y = \tan t$, $y(0) = 0$, $y'(0) = 1$.

This initial-value problem can be solved by the variation-of-parameter method, as follows. Note that the coefficients of the derivatives of y are all constants. To find the complementary solution, we solve the auxiliary equation $m^2 + 1 = 0$, which gives $m = \pm i$. Therefore, the complementary solution is $y = c_1 e^{ix} + c_2 e^{-ix} = c_1' \cos t + c_2' \sin t$ by Euler's Formula, where c_1, c_2, c_1', c_2' are constants, and the fundamental set of solutions is $\{\cos t, \sin t\}$. To find the particular solution, let $y_p(t) = u_1(t) \cos t + u_2(t) \sin t$. Then

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where W, W_1, and W_2 are the determinants (called *Wronskians*)

$$W = \begin{vmatrix} \cos t & \sin t \\ (\cos t)' & (\sin t)' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & \sin t \\ \tan t & (\sin t)' \end{vmatrix} \quad \text{and} \quad W_2 = \begin{vmatrix} \cos t & 0 \\ (\cos t)' & \tan t \end{vmatrix}$$

by Cramer's Rule. Therefore, we get

$$u_1'(t) = -\sin t \tan t = -\sec t + \cos t \quad \text{and} \quad u_2'(t) = \cos t \tan t = \sin t\,.$$

Integrating u_1' and u_2' gives

$$u_1(t) = -\ln(\tan t + \sec t) + \sin t \quad \text{and} \quad u_2(t) = -\cos t\,.$$

Thus, the particular solution is $y_p = -\cos t \ln(\tan t + \sec t)$. Since the general solution is the sum of complementary solution and particular solution,

$$y(t) = c_1' \cos t + c_2' \sin t - \cos t \ln(\tan t + \sec t)\,.$$

The constants can be determined from the initial values $y(0) = 0$ and $y'(0) = 1$, giving

$$y(t) = 2 \sin t - \cos t \ln(\tan t + \sec t)\,.$$

Knowing the solution of these differential equation, we now approximate the solution of the differential equation.

Example. Solve $y'' + y = \tan t$, $y(0) = 0$, $y'(0) = 1$ by Laplace approximation.

Applying Laplace transforms to both sides, the left-hand side becomes

$$\mathscr{L}\{y''+y\}(s) = \mathscr{L}\{y''\}(s) + \mathscr{L}\{y\}(s) = s^2 Y(s) - sy(0) - y'(0) + Y(s) = (s^2+1)Y(s) - 1\,.$$

Recall from (1) above that the approximation of Laplace transform of $\tan t$ is

$$\mathscr{L}\{\tan t\}(s) \approx \frac{1}{s^2} + \frac{2}{s^4} + \frac{16}{s^6} + \frac{272}{s^8}\,.$$

Therefore, we get the algebraic equation

$$(s^2 + 1)Y(s) - 1 = \frac{1}{s^2} + \frac{2}{s^4} + \frac{16}{s^6} + \frac{272}{s^8}\,.$$

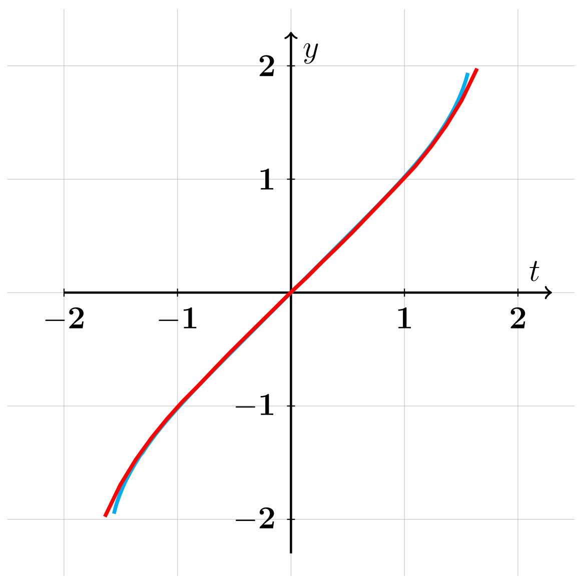
To find the solution to the differential equation, one needs to solve for $Y(s)$, and express the solution as partial fractions. Solving for $Y(s)$ gives

$$Y(s) = -\frac{257}{s^2} + \frac{258}{s^4} - \frac{256}{s^6} + \frac{272}{s^8} + \frac{258}{s^2+1}\,.$$

With the formula $\mathscr{L}^{-1}\{n!/s^{n+1}\} = t^n$, applying inverse Laplace transform to both sides gives the solution to the differential equation, which is

$$y(t) = \frac{17}{315}t^7 - \frac{32}{15}t^5 + 43t^3 - 257t + 258\sin t\,.$$

The approximated solution is very similar to the actual solution.



In the figure above, the red curve is the graph of $y(t) = 2\sin t - \cos t \ln(\tan t + \sec t)$, and the green curve is the graph of $y(t) = \frac{17}{315}t^7 - \frac{32}{15}t^5 + 43t^3 - 257t + 258\sin t$. The two curves are almost the same near $t = 0$.

6 Conclusion

In conclusion, Laplace transforms can be approximated and used to solve initial-value problems. While certain functions may not have a direct expression in terms of standard functions, the ability to approximate Laplace transforms remains a powerful tool in differential equations. Approximating solutions to differential equations, especially near $t = 0$, is very useful in computational applications such as solving differential equations for RC-circuits and pendulum oscillations.

References

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