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# Problems 1721–1730

*Parabola* would like to thank Toyesh Prakash Sharma for contributing Problem 1728.

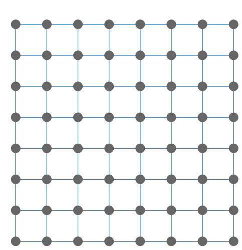
**Q1721** Nine people are participating in a "secret Santa" at an office Christmas party. Each brings along a gift which will be passed on to one of the people at the party, so that each person contributes one gift and receives one gift. The name of each person at the party is written on a slip of paper and placed in a bag: each person picks at random a slip from the bag. Then each person gives their gift to the person whose name they have drawn, who then passes it on to the person whose name *they* have drawn: for example, if Andy draws Betty's name and Betty draws Chiara's name, then Andy's gift ends up being given to Chiara. The idea behind passing gifts twice is to ensure that no–one will know who their gift originally came from.

The office newsletter editor, who is not much interested in secrecy, later publishes a report stating that gifts went from Andy to Chiara, from Betty to Harriet, from Chiara to Ivan, from David to Greg, from Elinor to Betty, from Frederica to Andy, from Greg to Elinor, from Harriet to David and from Ivan to Frederica.

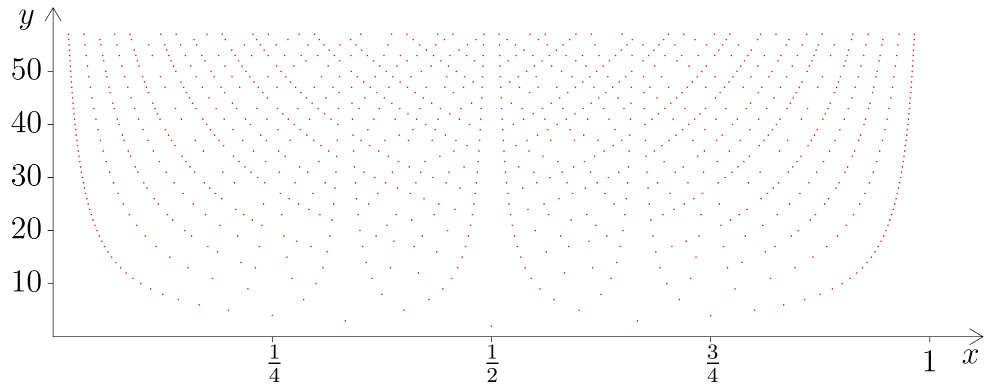
Prove that the report is wrong.

**Q1722** In Problem 1707, we considered all products of eleven different positive integers having sum $82$, and found the greatest common divisor (highest common factor) of all these products. Now change the sum to $s$, where $s$ is an integer not less than $66$. (If $s < 66$, then there is no collection of eleven different positive integers with sum $s$, and so the problem does not make much sense.) Find the *smallest* value of $s$ for which the greatest common divisor of all the corresponding products of eleven numbers is $1$. If $s_{\min}$ is this smallest value and we consider a sum $s > s_{\min}$, does it necessarily follow that the greatest common divisor of all products of eleven different positive integers with sum $s$ is still $1$?

**Q1723** In how many ways can one select $5$ points from the $64$ shown, such that at least three of the chosen points lie in a straight line? Here a "straight line" means one of the horizontal or vertical lines shown in the diagram, and the points in a line *do not* need to be adjacent points on the grid.

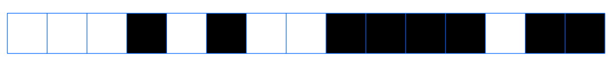


**Q1724** If $x$ is a rational number, then we define $f(x)$ to be the denominator of $x$. That is, if $x = p/q$ in lowest terms, then $f(x) = q$. Part of the graph of $y = f(x)$ is shown below. Can you explain the "dotted curves" appearing in the image? Or any other notable features?




**Q1725** Let $n$ be a positive integer. Prove that the square $n^2$ has no factor $m$ in the interval $n < m \le n + \sqrt{n}$.

**Q1726** Given a positive integer $n$, we seek sequences $a_1, a_2, \ldots, a_k$ of one or more positive integers for which it is possible to arrange a row of black and white squares such that the black squares occur in blocks of length $a_1, a_2, \ldots, a_k$, in that order from left to right, and there is at least one white square between adjacent blocks. For example, if $n = 15$, then one of the possible sequences is $1, 1, 4, 2$, as shown by the following row.



For a given positive integer $n$, in how many ways can this be done?

This problem was inspired by the "nonograms" puzzle which can be found on various websites.

**Q1727** Is it possible to find a square number beginning with any given sequence of digits?

**Q1728** Show that if $x, y, z$ are positive real numbers and $xyz = 1$, then

$$x^{1/2} + y^{1/4} + z^{1/6} \ge 2^{2/3} 3^{1/2} .$$

**Q1729** We have $n$ coins, all placed heads up on a table. It is permitted to select any $k$ of the coins and flip them; and to do a similar operation repeatedly. Here, $k$ is a fixed positive integer less than $n$. The aim is to get all of the coins facing tails up. Prove that this can be done if and only if either $n$ is even or $k$ is odd.

**Q1730** It is well known that to trisect an arbitrary angle, using ruler and compasses in the classically permissible manner, is impossible. However, the job can be done by origami!

Let $OW$ be one side of a rectangular sheet of paper, and make a fold $OZ$ so that $\angle WOZ$ is the angle we wish to trisect. Make two equally spaced folds $AX$ and $BY$ parallel to $OW$, as shown in the diagram. Fold the corner $O$ back into the page in such a way that $O$ lies on the line $AX$, at a point we call $C$, and $B$ lies on $OZ$, at a point $D$. Prove that $\angle WOC$ is one third of $\angle WOZ$.