O200 The hole in the head of a spanner is a regular hexagon with sides of length a. The spanner is used to tighten a square nut with sides of length b. Find conditions satisfied by a and b for this to be possible.

SOLUTIONS

Solutions to Problems 181–180 in Vol 8 No 2

The names of successful problem solvers appear after the solution to Problem 190.

Junior

J181 Show that the product of 4 consecutive integers is always one less than a perfect square.

Answer

$$\begin{aligned}(n-1).n.(n+1).(n+2) &= [n.(n+1)].[(n-1).(n+2)]\\ &= (n^2+n).(n^2+n-2)\\ &= [(n^2+n-1)+1].[(n^2+n-1)-1]\\ &= (n^2+n-1)^2-1\end{aligned}$$

J182 Find the integral solutions of the equation $y^3-x^3=91$.

Answer $\quad y^3-x^3 = (y-x)(y^2+xy+x^2) = 91.$

Hence $\quad y-x = \pm 1, \pm 7, \pm 13,$ or ± 91

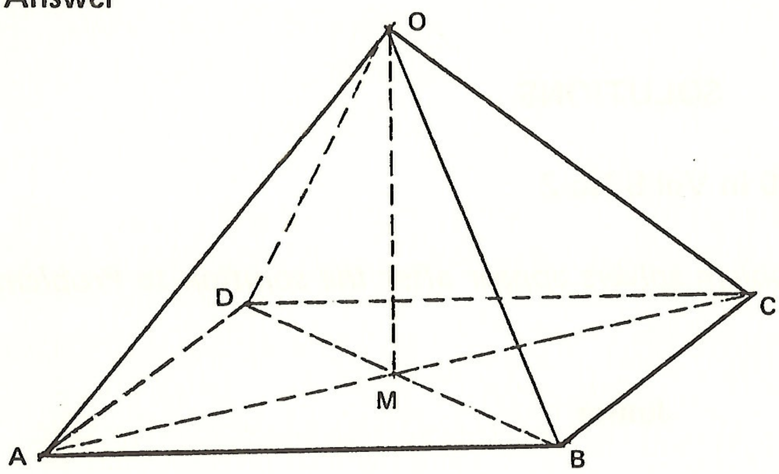
and $\quad y^2+xy+x^2 = \pm 91, \pm 13, \pm 7,$ or ± 1 respectively.

Since $(y+x)^2 = 1/3[4(y^2+xy+x^2)-(y-x)^2]$ must be positive, the values -91, -13, 7, -7, 1 and -1 of (y^2+xy+x^2) must be discarded. Corresponding to $y-x=1$, we obtain $y+x=\pm 11$, yielding solutions $(x,y) = (5,6), (-6,-5)$.

Corresponding to $y-x=7$, we obtain $y+x=\pm 1$, yielding the solutions $(x,y) = (-3,4)$ and $(-4,3)$.

J183 Four equal spheres are placed on a horizontal plane so that each touches two others. A fifth equal sphere, rests on top of the other four. If the radius of a sphere is r, find the height of the highest point of the fifth sphere above the horizontal plane.

Answer



In the diagram the vertices of the square ABCD are the centres of the 4 spheres resting on the table, and O is the centre of the sphere on top. The lengths of the sides of the square and of OA, OB, OC and OD are all 2r, where r is the radius of the spheres. By the symmetry of the figure, the perpendicular OM from O to the plane of the square has its foot M at the intersection of the diagonals of the square. Hence, the length of $AM = \sqrt{2}\, r$. Now applying Pythagoras' theorem to the right-angled triangle OAM,

$$OA^2 = AM^2 + OM^2$$

$$4.r^2 = 2.r^2 + OM^2$$

Hence $$OM = \sqrt{2}r.$$

Since all of A, B, C and D, and therefore also M is at a height r above the table, the height of O above the table is $(1 + \sqrt{2}).r$, and the height of the top of the upper sphere is $(2 + \sqrt{2}).r$.

Open

O184 Let a, b, c and d be any four positive integers. Let a_1, b_1, c_1 and d_1 be the differences between a and b, b and c, c and d, and d and a respectively. The same process is then used to obtain a_2, b_2, c_2 and d_2 (e.g. $a_2 = |a_1 - b_1|$) and so on. Show that eventually four zeros must be obtained.

For example, starting with

7 24 32 32

one obtains in succession

17	8	0	25
9	8	25	8
1	17	17	1
16	0	16	0
16	16	16	16
0	0	0	0

Answer Note first that the largest of a_1, b_1, c_1 and d_1 is not greater than the largest of a, b, c and d. In fact

$$\max(a_n, b_n, c_n, d_n) \leqslant \max(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}) \text{ for all } n \geqslant 1.$$

It is also obvious that if a_n, b_n, c_n and d_n all have a common factor k (say, $a_n = k.A_n$; $b_n = k.B_n$ etc), the set of differences a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1} is k times the corresponding set of differences A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}. Thus it is clear that a_n, b_n, c_n, d_n will yield eventually four zeros if and only if A_n, B_n, C_n, D_n do so. That is, we may allow ourselves the additional operation of cancelling any common factor of the four numbers obtained at any stage.

Now observe that after at most four stages, a set of four even numbers is obtained. [For example, if there are initially 3 even numbers and 1 odd, one obtains an arrangement such as

E	E	E	O
E	E	O	O
E	O	E	O
O	O	O	O
E	E	E	E

The columns in such a scheme may be permuted cyclically, so the position of the O in the first line is unimportant. The only arrangement (apart from cyclic permutation) which does not appear in this example is O O O E which would yield the same second line.]

If we denote by A_n, B_n, C_n, D_n the four numbers obtained after n stages of the process *and* after any common factor has been cancelled out, it follows that

$$\max(A_{n+4}, B_{n+4}, C_{n+4}, D_{n+4}) \leqslant \tfrac{1}{2} \max(A_n, B_n, C_n, D_n);$$

and
$$\max(A_{4k}, B_{4k}, C_{4k}, D_{4k}) \leqslant 1/2^k \max(A_0, B_0, C_0, D_0) < 1 \text{ for k sufficiently large.}$$

Hence we must have $\max(A_{4k}, B_{4k}, C_{4k}, D_{4k}) = 0$ for k sufficiently large, and this completes the proof.

Note: If instead of 4 numbers, we start with m numbers, it is not necessarily true that 4 zeros are eventually obtained. For example if m = 3, or 5, and we start with (1,1,0) and (1,1,1,1,0) respectively, you may easily check that the resulting schemes cycle. Only when m is a power of 2 is a set of zeros necessarily obtained.

0185 Given the sequence of numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . . where each number (beginning with the third) is the sum of the 2 preceding numbers; show that there exists a number amongst the first 100,000,001 terms of this sequence terminating with four zeros.

Answer Let n be any positive integer, and replace each term in the given Fibonacci sequence by the least non-negative remainder when it is divided by n. (For example, if n is 6, we get

0, 1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1, 0, 1 . . .

Given any 2 consecutive remainders r_k and r_{k+1}, it is easy to calculate both r_{k+2} and r_{k-1}. Hence the sequence of remainders is completely determined in both directions by 2 neighbouring terms. Since there are only n different remainders, and therefore at most n^2 different pairs of remainders, it is easy to see that the sequence must eventually cycle with a "period" of length at most n^2. Because the sequence is determined backwards as well as forwards by 2 consecutive terms, the recurring behaviour must start right from the beginning. As the first term is 0, it now follows that 0 must occur again infinitely often; i.e. there are infinitely many terms in the original sequence divisible by n, in fact, at least 1 in any block of length n^2.

0186 Solve the equation

$$\sqrt{a - \sqrt{a + x}} = x.$$

Answer After squaring, rearranging and squaring again, one obtains

$$x^4 - 2ax^2 - x + a^2 - a = 0 ......(1).$$

This, being of degree 4 in x, does not seem very easy to solve. However it is only of degree 2 in a: $a^2 - (2x^2 + 1).a + (x^4-x) = 0$; and the formula for its solution gives

$$a = \tfrac{1}{2}[(2x^2 + 1) \pm \sqrt{[(2x^2 + 1)^2 - 4(x^4-x)]}]$$
$$= \tfrac{1}{2}[(2x^2 + 1) \pm (2x + 1)].$$

Hence either $x^2 + x + (1-a) = 0$; or $x^2 - x - a = 0$.
These give $x = \tfrac{1}{2}(-1 \pm \sqrt{(4a-3)})$......(2); or $x = \tfrac{1}{2}(1 \pm \sqrt{(4a + 1)})$......(3).
Since the same equation (1) is also obtained from any of the 4 surd equations $\pm\sqrt{a} \pm \sqrt{(a + x)} = x$, the admissibility of the solutions (2) and (3) must be examined. By substitution it is not difficult to check that $x = \tfrac{1}{2}(-1 + \sqrt{(4a-3)})$ is the only solution when $a \geqslant 1$.
It is clear than when $a < 1$ there is no real solution.

0187 (i) How many roots has the equation

$$\cos x = \frac{x}{50} ?$$

(ii) How many roots has the equation

$$\cos x + \log_e x = 0?$$

Answer *(i).* The stationary points of the function $f(x) = \cos x - x/50$ occur at solutions of $f'(x) = -\sin x - 1/50 = 0$. These are the points $x = -0.0200013... + 2k\pi$ (1) and $x = +0.020... + (2m + 1)\pi$, (2), where k and m stand for any integers. At these points $f''(x) = -\cos x = \pm 0.9998...$, the + applying to the values (1), and the – to the values (2). Hence, the points (1) are minima of f(x), and (2) are maxima. The value of f(x) at a minimum is negative provided $\cos x - x/50 = -0.9998... -x/50 \leqslant 0$, i.e. if $x \geqslant -49.99...$ which is true in formula (1) provided $k \geqslant -7$, but false if $k < -8$. Similarly, the value of f(x) at a maximum (2) is positive provided $m \leqslant 7$, but negative if $m > 8$.

Since there is one zero of f(x) between a minimum and a neighbouring maximum provided the function changes sign, the above information enables us to count 31 solutions of the equation $f(x) = 0$, the smallest of them lying between the positive maximum (2) with $m = -8$ and the negative minimum (1) with $k = -7$, the largest lying between the maximum (2), $m = 7$, and the minimum (1), $k = 8$.

(ii). If $x > e$, $\log_e x > 1$, so $g(x) = \cos x + \log_e x > 0$.

If $x \leqslant 0$, $\log_e x$ is undefined. Hence all the zeros of $g(x)$ lie in the range $0 < x \leqslant e$.

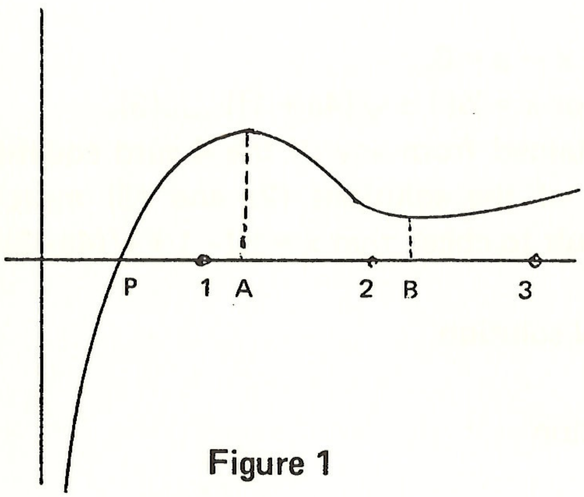


Figure 1

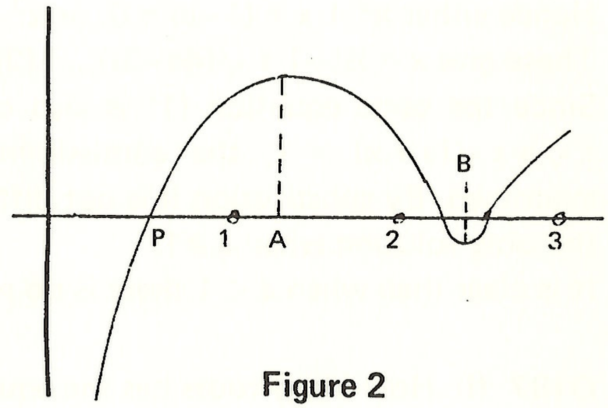


Figure 2

The approximate behaviour of $g(x)$ in this range is easy enough to determine. As $x \to 0+$, $\log_e x$ and therefore also $g(x) \to -\infty$.

Note that $g'(x) = -\sin x + 1/x$, which is obviously positive for $0 < x \leqslant 1$, but negative for $x = \pi/2 = 1.57\ldots$. Hence there is a maximum of $g(x)$ somewhere in $1 < x < 1.57\ldots$; (the point A in the diagrams). Since both $\log_e x$ and $\cos x$ are positive in $1 < x \leqslant \pi/2$, $g(x)$ is certainly positive at A, and there must be one zero, P, of $g(x)$ in $0 < x < 1.57\ldots$. For $x > \pi/2$, $g'(x)$ again becomes positive by $x = \pi$, so there is a minimum of $g(x)$ at some point B. If the value of $g(x)$ is positive at B (as in Figure 1) then P is the only zero of $g(x)$. The other possibility has $g(x) \leqslant 0$ at B, resulting in further zeros of $g(x)$ near B. However, some calculation shows that at B, $x = 2.77\ldots$ (solve the equation $g'(x) = -\sin x + 1/x = 0$ by a numerical method, or by using appropriate tables) and at B, $g(x) = \cos 2.77 + \log_e 2.77 = -.933 + .986$ which is positive. Hence Figure 1 applies, and there is only one root of the equation.

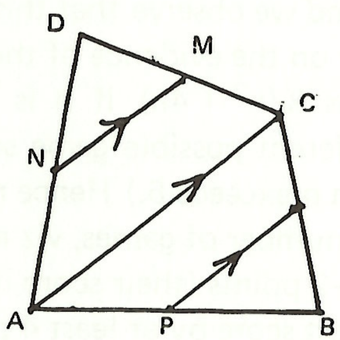
0188 The following excellent problem appeared in the 1967 Tasmanian Schools Mathematics Competition and we reprint it by courtesy of the Mathematical Association of Tasmania. It was not solved by the competitors.

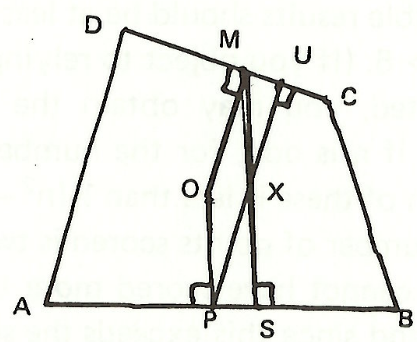
A pupil is situated at the centre of a circular swimming pool. A non-swimming teacher (who wishes to administer punishment) waits for him at the edge. He can run 4 times as fast as his quarry can swim, but cannot catch him on land. Has the pupil a strategy by which he can escape from the pool and elude capture?

Answer Denote the swimmer's position by A, the runner's by B, the centre of the pool by C, the radius of the pool by r, and the distance CA by d. Provided $d < r/4$, the swimmer is able to keep C on the straight line between himself and B (since his maximum angular velocity about C is v/d where v is his speed, and this is greater than 4v/r, the runner's maximum angular velocity about C). Hence he cannot be prevented from reaching a point at a distance kr from C, where $\frac{1}{4}(4-\pi) < k < \frac{1}{4}$, with BCA straight.

From this point his obvious strategy is successful; i.e. he swims along a radius to the edge of the pool, a distance $D = (1-k).r$. Since $\frac{1}{4}(4-\pi) < k$, $1-k < \frac{1}{4}\pi$, so that $4D < \pi r$, which is the distance the runner must cover to catch him. Hence he is able to escape.

0189 ABCD is a cyclic quadrilateral and M, N, P, Q are the midpoints of sides CD, DA, AB, BC respectively. MS, NT, PU, QV are perpendicular to AB, BC, CD and DA respectively. Prove that they are concurrent.





Answer MNPQ is a parallellogram, since MN and PQ are each parallel to AC and half its length. Hence the mid-point of MP coincides with the mid-point of NQ. [1]. Let O be the centre of the circle through ABCD. Let PU and MS intersect at X. Since OP ⊥ AB, OP ∥ MX. Similarly, OM ∥ PX, so that OPXM is a parallelogram. Hence the mid-point of OX is also the mid-point of MP. A similar argument shows that if the intersection of NT and QV is Y then the mid-point of OY coincides with the mid-point of NQ. Using [1], we see that OX and OY have the same mid-point, whence X coincides with Y.

O190 In a round robin soccer tournament (i.e. each team plays every other team) no games are goal-less, no two results are identical and the number of goals scored in any one game does not exceed the number of teams. Two points are awarded for a win and 1 point each for a draw. The difference of the point scores of the first two teams is equal to four times the difference of the point scores of the last two teams. No two teams emerged with the same number of points. How many teams were entered in the competition?

Answer If 2k goals are scores in a game, the possible results are (2k v 0); (2k–1 v 1); (2k–2 v 2); . . .; (k v k); a total of k+1 different possible results. If 2k+1 goals are scores the possible scores are (2k+1 v 0); (2k v 1); . . . ; (k+1 v k); again k+1 different possible results. If n teams participate in the tournament, the number of different results possible under the stated conditions may be found by summing the above results when the number of goals scores is 1, 2, 3, . . . , n. That is, the first n terms of the series 1 + 2 + 2 + 3 + 3 + 4 + . . . must be added. This gives 3, 5, 8, 11, 15, 19 for n = 2, 3, 4, 5, 6, 7 respectively. The number of games to be played is $^nC_2 = \frac{1}{2}.n.(n-2)$ whose value is 1, 3, 6, 10, 15, 21 . . . for n = 2, 3, 4, 5, 6, 7 . . . The conditions laid down require that the number of different possible results should be at least $^nC_2$, and we observe that this ceases to be true for $n > 6$. (If you object to relying merely on the evidence of the tables of values calculated, you may obtain the formulas $\frac{1}{4}(n^2 + 4n)$ if n is even, and $\frac{1}{4}(n^2 + 4n-1)$ if n is odd, for the number of different possible game scores, and show that each of these is less than $\frac{1}{2}(n^2-n)$ when n exceeds 6.) Hence $n \leqslant 6$. [1]

The total number of points scored is twice the number of games, viz n^2-n. The winning team cannot have scored more than 2n–2 points (their score if they win every game), and since this exceeds the second best score by at least 4 points, the maximum numbers of points that could be scored by the other n–1 teams are 2n–6, 2n–7, 2n–8, . . . , 2n–*(n+4)*. Hence we must have $(2n-2) + (2n-6) + (2n-7) + \ldots + (2n-n-4) \geqslant n^2-n$, which simplifies to $n^2-7n+6 \geqslant 0$. Hence $n \geqslant 6$.

Combining this with [1], we see that n = 6, if any solution is possible. A possible score card of the tournament is

A	A	B	C	D	E	F	total
A		2	2	2	2	2	10
B	0		2	0	2	2	6
C	0	0		1	2	2	5
D	0	2	1		0	1	4
E	0	0	0	2		1	3
F	0	0	0	1	1		2

Note that when n = 6 there are only 15 different game scores possible, three of which are draws, and as 15 games are played, all three draws must have occurred.

☆ ☆ ☆

Some Mental Arithmetic Short-cuts

(a) To square any number ending in 1, add and subtract 1 from the number, multiply the two resulting numbers and then add the square of 1. You can make up similar rules for squaring other numbers in a similar way. For example, $23^2 = 20 \times 26 + 3^2 = 529$.

(b) See if you can make up a similar trick for squaring fractions.

(c) If you are given two numbers where the tens are the same and the units add up to ten (say ab multiplied by ac), write down a (a + 1) and then write down b c next to it. For example, 67 x 63 = 4221 (since 6 x 7 = 42, 7 x 3 = 21).

(d) If you are asked to multiply a number just less than 1000 by another three digit number (say 1000–a and b where a is small), subtract a from b and b from 1000 and then write down b–a and a x (1000–b). For example, if the numbers are 996 and 858, we write down 858 – 4 = 854 and then 4 x (1000–858) = 4 x 142 = 568. So the answer is 854,568.

Try these out on your friends and make them think that you are very fast at mental arithmetic.

Solvers of Problems 181–190 (Vol 8 No 2)

Seppo Ahlstedt (Dickson High, ACT) 181-2.
Graham Beirman (Sydney Grammar) 190.
John Christodoulou (East Hills Boys High) 188.
Henri de Feraudy (St Joseph's College, Hunter's Hill) 184-6, 188.
Alan Fekete (Sydney Grammar) 181.
Rosario Filippello (Sydney Boys High) 188.
John Harvey (Narrabundah High) 188.
Graeme Henderson (Normanhurst Boys High) 185, 188.
Tony Holzherr (Cabramatta High) 188.
Nicholas Pipe (East Hills Boys High) 181.
A. Pollack (Sydney Grammar) 190.
Robert Smith (Endeavour High) 181, 183.
Chris Wood (Newington College) 186-7.
Richard Wood (Newington College) 188.
R. Yager (Normanhurst Boys High) 186.
Michael Young (Caringbah High) 188.

So far there have been no correct solutions for 189. A few misread the question, believing that perpendiculars were to be erected at the mid-points of the sides of the given cyclic quadrilateral, when of course they were all concurrent at the centre of the circumcircle.

☆ ☆ ☆

Answers

A Short Competition: Since 68 students have to be eliminated, 68 matches will be needed.

The Power of Digits: 9^{21} = 109,418,989,131,512,359,209 (9^{22} has 21 digits).

Factors and Factors: The easiest answer is $m = 1$, $n = 11! + 1$.