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MOTION OF PLANETS AND SATELLITES

Section 1. Kepler's Laws

Early in the 17th century, Johannes Kepler established, from actual observations of the positions of the planets in the sky, three laws of planetary motion. All three laws were based on the, then very controversial, assumption of Copernicus that planets revolve around the sun. Copernicus had assumed that the motion of the planets about the sun was in circles, and as a result his description of the actual motion of the planets was not significantly better than that of the ancient Greeks. This was one reason, but by no means the only one, for the lack of acceptance of his idea.

Kepler, basing himself on observations by the astronomer Tycho Brahe, established a different set of laws, which are now known as the three laws of Kepler. They are:-

1. Each planet moves around the sun in an ellipse with the sun at one of the two foci of that ellipse.

2. The motion of each planet along its orbit is such that equal areas are swept out in equal times (this will be explained shortly).

3. The time each planet takes to go once around its ellipse is proportional to the 3/2 power of the long axis of that ellipse.

The second of these laws needs some explanation. Let us look at Figure 1, which shows schematically the elliptical path of a planet. At point A the planet is moving with a vector velocity $\vec{v}$.

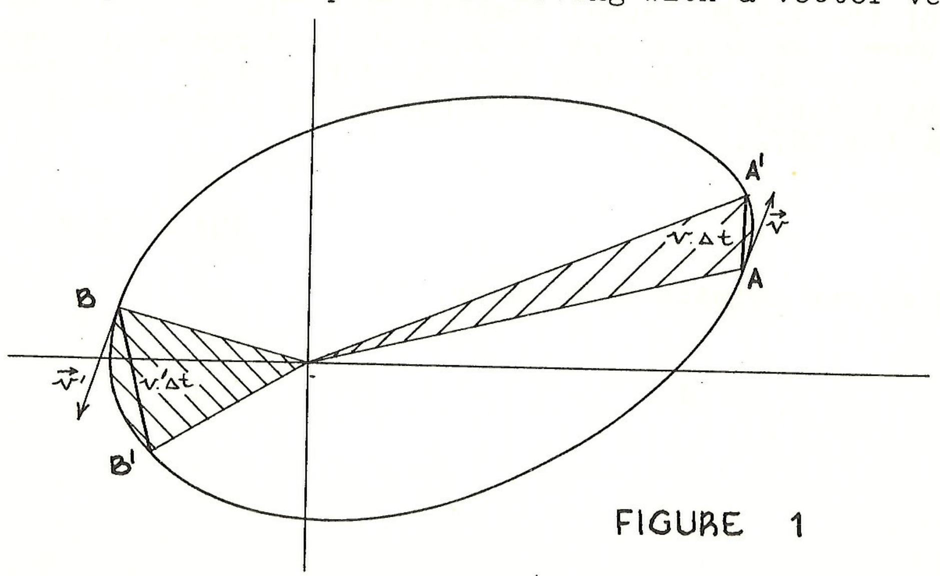

FIGURE 1

In a short time interval Δt, the distance the planet moves along its path is $v \Delta t$, as indicated on the figure. If we draw straight lines from point A to the sun, which we have placed at the origin of the system of coordinates, as well as from the point A' which the planet reaches after time Δt, then these two straight lines together with the segment AA' form a little triangle, whose area is shaded on the figure. Let us now look at another point on the orbit of the planet, point B. This point is closer to the sun and it turns out that this means that the velocity $\vec{v}'$ at that point is much bigger. Correspondingly the distance $v' \Delta t$ between points B and B' on the path of the planet is bigger than the distance from A to A'. Drawing straight lines to the sun once more, we get the other shaded triangle shown in Figure 1. Now we can understand the statement of Kepler's second law. It is that the two shaded areas are equal if the two time intervals Δt are equal to each other.

In this article, using no more than the mathematics that you know and Newton's laws of motion, I intend to derive the second and the third laws of Kepler. It turns out that Kepler's first law, the one which says that the orbit is an ellipse, is harder to derive, and would not fit within the space of this article.

Section 2. Polar Coordinates

Before we can discuss the motion of planets we need to develop a bit of elementary trigonometry for the motion of a point around a centre. Let us refer to Figure 2. The point A in this figure has

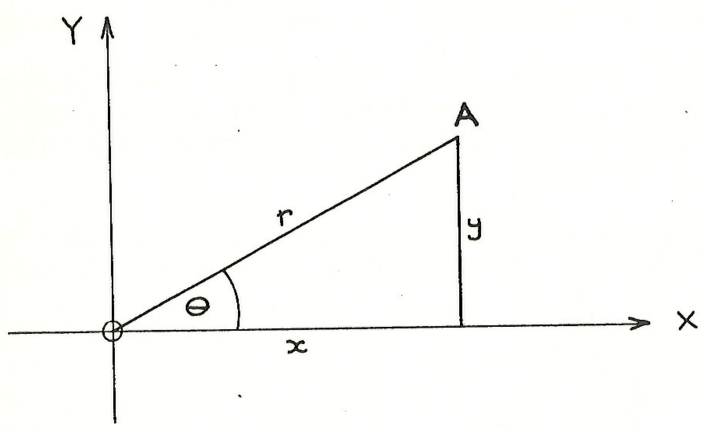

FIGURE 2.

coordinates x and y as its ordinary Cartesian coordinates. However, if we are interested in the motion of a planet around a centre of attraction such as the sun, then a better way to define the position of a point in the plane is through its distance from the origin, shown as r in the figure, together with the angle θ. Using ordinary trigonometry on the triangle OAB, we immediately get

(1) $\frac{x}{r} = \cos\theta$

(2) $\frac{y}{r} = \sin\theta$

We can solve these equations for x and y, respectively, to get

$$(3) \quad x = r \cos\theta$$

$$(4) \quad y = r \sin\theta$$

In these equations, all quantities, r, θ, x, and y, are functions of the time t.

In order to get the velocity of a point moving in the xy-plane, we differentiate x and y with respect to the time. Using the ordinary laws for differentiation of the product of two functions, as well as of a function of a function, we obtain equations 5 and 6 for the x- and y-components of the velocity, respectively.

$$(5) \quad v_x = \frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$$

$$(6) \quad v_y = \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$$

Since Newton's second law of motion involves, not the velocity, but the acceleration, we shall need to differentiate once more with respect to the time, so as to get the x- and y-components of the acceleration. These are shown below:

$$(7) \quad a_x = \frac{dv_x}{dt} = \frac{d^2r}{dt^2}\cos\theta - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}$$

$$(8) \quad a_y = \frac{dv_y}{dt} = \frac{d^2r}{dt^2}\sin\theta + 2\cos\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}$$

These two equations are <u>not</u> obvious, and it takes a few minutes of work to obtain them. However, their derivation is a straight-forward application of the laws of differentiation of products and differentiation of a function of a function.

Let us now go to Figure 3 and let us take a look at the vector $\vec{u}$ connecting the points A and C. This vector is not meant to be

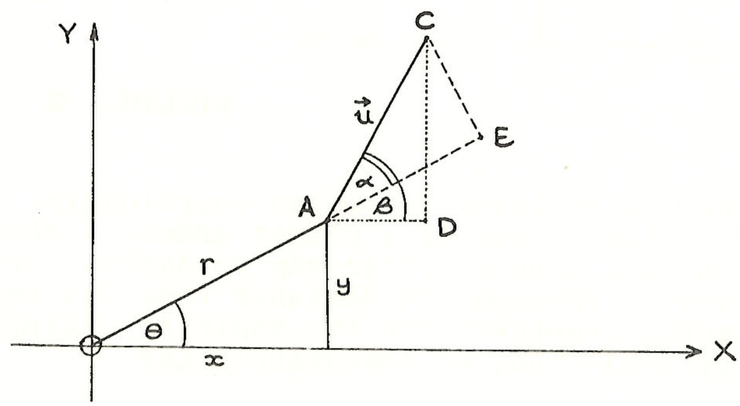

FIGURE 3

any particular vector quantity. You may think of it as a velocity vector or an acceleration vector, or any other vector. Normally, we describe this vector in terms of x and y components. The x-component of the vector $\vec{u}$ is the distance AD on the figure, the y-component is the distance DC. Using trigonometry on the triangle ADC in the ordinary way, we obtain equations 9 and 10 for these two components of the vector $\vec{u}$.

(9) $u_x = u \cos\beta$

(10) $u_y = u \sin\beta$

In these two equations, u is the magnitude of the vector quantity $\vec{u}$, i.e. the actual distance from A to C on the figure, and the angle β is as indicated on the figure.

However, when we are dealing with motion around a centre of attraction, these two components, i.e. the ordinary Cartesian components of a vector quantity $\vec{u}$, are not the most convenient ones to use. Also indicated on Figure 2 is another triangle, AEC, which can also be used to describe the vector quantity $\vec{u}$. We shall call the distance from A to E the "r-component of $\vec{u}$" u_r, since it is the component in the direction in which the quantity r would be increasing. The distance from point E to point C we shall call the "θ-component of the vector u", because that is the direction in which the point A would move if the coordinate r stays constant and the angle θ increases. Using trigonometry on the triangle AEC and the angle α indicated on Figure 2, we get equations 11 and 12 for these alternative components of the vector $\vec{u}$.

(11) $u_r = u \cos\alpha$

(12) $u_\theta = u \sin\alpha$

Since the set of components (9) and (10) and the set of components (11) and (12) describe one and the same vector, there must be a relationship between them. Looking at the triangles in figure 2 the following relation between the angles is obvious.

(13) $\alpha = \beta - \theta$

Let us substitute equation (13) into equations (11) and (12), and let us make use of the formula for the cosine and the sine of the difference of two angles. The result is

(14) $u_r = u \cos\beta \cos\theta + u \sin\beta \sin\theta$

(15) $u_\theta = u \sin\beta \cos\theta - u \cos\beta \sin\theta$

When we use equations (9) and (10) within the right hand sides of equations (14) and (15), we obtain the desired relationship between the two different types of component of one and the same vector:

(16) $$u_r = u_x \cos\theta + u_y \sin\theta$$

(17) $$u_\theta = u_y \cos\theta - u_x \sin\theta$$

So far, the vector u has been just any arbitrary vector in the plane. Let us now, however, apply formulas (16) and (17) to the velocity vector and to the acceleration vector. When we substitute the x and y components of the velocity vector, equations (5) and (6), into equations (16) and (17), we obtain:

(18) $$v_r = v_x \cos\theta + v_y \sin\theta = \frac{dr}{dt}$$

(19) $$v_\theta = v_y \cos\theta - v_x \sin\theta = r\frac{d\theta}{dt}$$

Before going on, let us observe that equation (19) gives us an alternative, and more useful, way of describing Kepler's second law of planetary motion. Let us refer now to Figure 4. The shaded area,

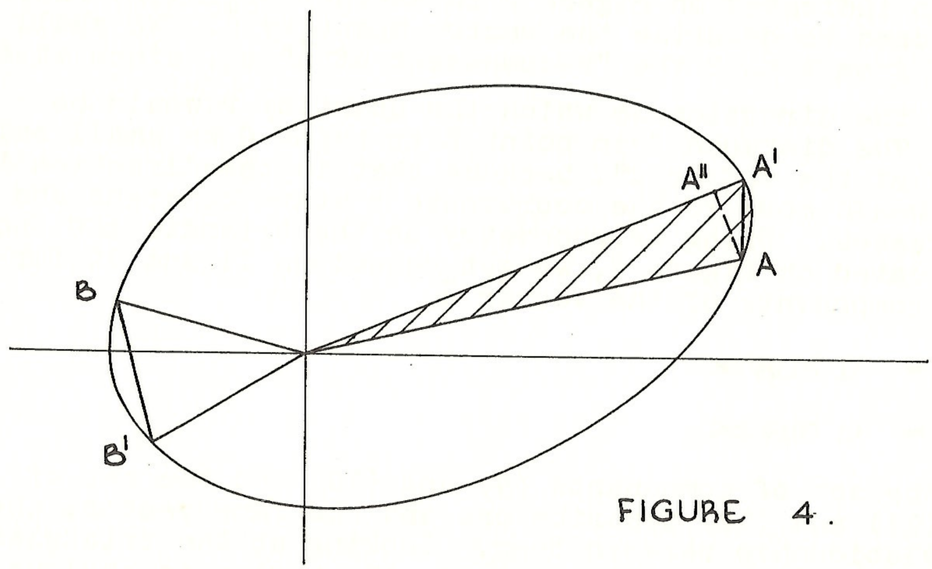

FIGURE 4.

OAA", on this figure which appears in Kepler's second law, has been divided into two separate areas by putting a point A" into the figure, such that the angle OAA" is a right angle. If the time interval Δt is sufficiently short, then the distance from A to A" is the product of the component of the velocity in that direction, i.e. v_θ, and the time Δt. The area of the triangle OAA" is half the base OA, which is the distance we have called r, times the height of the triangle AA". Thus the shaded area OAA" is given by:

(20) $$\text{Area } OA A'' = \tfrac{1}{2} r (v_\theta \Delta t) = \tfrac{1}{2} r^2 \frac{d\theta}{dt} \Delta t$$

In the last part of this equation we have made use of equation (19) for v_θ. The other area AA'A" of Figure 1 is negligible by comparison, being of order $(\Delta t)^2$.

Kepler's second law states that the area (20) is the same no matter where we are on the path of the planet, provided only the time interval Δt is the same. Thus an alternative statement of Kepler's second law is that the quantity L defined by

$$L = r^2 \frac{d\theta}{dt} = \text{Constant} \tag{21}$$

is constant during the motion of the planet. It is in this form that we shall derive Kepler's second law.

Having discussed the velocity vector, let us now turn to the acceleration vector. We again use equations (16) and (19) to get the r and θ components of the acceleration vector, starting from the x and y components given in equations (7) and (8), respectively. The result is:

$$a_r = a_x \cos\theta + a_y \sin\theta = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \tag{22}$$

$$a_\theta = a_y \cos\theta - a_x \sin\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} \tag{23}$$

Though these equations may look somewhat formidable, you are actually already quite familiar with one special case of these equations, namely the special case of uniform circular motion. In uniform circular motion, the quantity r doesn't change, and the relation between the angle θ and the time is given by:

$$r = \text{Constant}, \quad \theta = \omega t \qquad [\text{UNIFORM CIRCULAR MOTION}] \tag{24}$$

where ω is a constant. Let us now substitute these special results for uniform circular motion into our general results for the velocity and for the acceleration. Substitution into equations (18) and (10) yields

$$\left.\begin{aligned} v_r &= 0 \quad &(25)\\ v_\theta &= r\omega \quad &(26)\end{aligned}\right\} [\text{UNIFORM CIRCULAR MOTION}]$$

and substitution into equations (22) and (23) yields

$$\left.\begin{aligned} a_r &= -r\omega^2 = -\frac{v^2}{r} \quad &(27)\\ a_\theta &= 0 \quad &(28)\end{aligned}\right\} [\text{UNIFORM CIRCULAR MOTION}]$$

These last two equations are the famous centripetal acceleration about which we have heard already. The acceleration vector has zero component in the θ direction, meaning therefore that the vector is along the line connecting the particle moving in uniform circular motion with the centre of the circle. The minus sign in equation (27) means that the acceleration vector points towards the centre of the circle, and the value $r\omega^2$ of this acceleration vector can be related to the velocity through equation (26) to be equal to v^2/r, which is the result that may have puzzled you in the past. It is now apparent that these results are merely special cases of our general results (22) and (23).

Section 3. Central Forces and Kepler's second Law

We are now in a position to derive Kepler's second law of motion from the results which we have obtained. The essential step here is application of Newton's second law of motion, relating the vector force $\vec{F}$ acting on a particle with its mass m and its acceleration vector $\vec{a}$. This relation is

(29) $$\vec{F} = m\vec{a}$$

If we use r and θ components of the vectors on both sides of equation (29), we then obtain:

(30) $$F_r = m a_r$$

(31) $$F_\theta = m a_\theta$$

Simple as equations (30) and (31) may appear, a very important result is already sitting in equation (31).

If, as is natural for the motion of a planet around the sun, we place the sun at the centre of our coordinate system, then the gravitational force between the planet and the sun is in the direction from the planet to the sun. That is, the only non-zero component of the vector $\vec{F}$ is the component F_r. The other component, at rightangles to this direction, i.e. the component F_θ, is zero. Thus, we obtain for any central force:

(32) $$a_\theta = 0 \qquad [\text{CENTRAL FORCE}]$$

Substitution of equation (23) into (32) gives

(33) $$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0$$

Let us now go back to Kepler's second law in the form of equation (21). If the quantity L is indeed a constant of the motion, then its time derivative must be equal to zero. Let us therefore differentiate L with respect to the time. This differentiation is shown in equation (34).

(34) $$\frac{dL}{dt} = \frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right] = 2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2}$$

By comparing equations (33) and (34) we immediately get the result:

(35) $$\frac{dL}{dt} = 0$$

The time derivative of L is zero, so that <u>L stays constant during the motion of the planet around the sun. We have therefore derived</u>

Kepler's second law.

Observe the general nature of this derivation. The validity of Kepler's second law does not depend upon the way in which the magnitude of the force F depends upon the distance r between the body and the centre of attraction. We have not yet made any use at all of Newton's law of gravitation, that the force is inversely proportional to the square of the distance r. Kepler's second law is valid no matter how that force depends upon r. Kepler's second law is a consequence of the fact that the force points towards the centre of attraction, i.e. that it is a central force. That, and only that, is what we have used in this derivation.

Section 4. How Might Newton have Guessed his Inverse Square Law?

When I was a student in high school I was always puzzled by how Newton could possibly guess such a law as his inverse square law. Of course, I don't really know even now how he did. But it is not unreasonable to make a guess.

It is true that the actual motion of planets around the sun is along an elliptical orbit. However, the ellipses are by no means as eccentric, that means as different from a circle, as the one shown for purposes of illustration in Figure 1. Actual planets revolve around the sun in very nearly circular orbits. Let us therefore for the moment assume that we have some ideal planet, different from the actual ones, which revolves around the sun with uniform circular motion. We can then apply equations (27) and (28). Substitution of equation (27) into equation (30) yields the result

$$(36) \qquad F_r = ma_r = -\frac{mv^2}{r} \qquad \text{[UNIFORM CIRCULAR MOTION]}$$

Now let us look at Kepler's third law. The time for a planet to run once around its assumed circular orbit, let us call this time T, is the circumference of this orbit divided by the speed. This gives:

$$(37) \qquad T = \frac{2\pi r}{v} \qquad \text{[UNIFORM CIRCULAR MOTION]}$$

We solve equation (37) for v, and substitute into equation (36) to get:

$$(38) \qquad F_r = -\frac{4\pi^2 mr}{T^2} \qquad \text{[UNIFORM CIRCULAR MOTION]}$$

At this stage, let us use Kepler's third law, which says that the square of the period, T^2, is proportional to r^3, the cube of the radius of the circle. (Kepler of course was talking about the long axis of an ellipse, but if the ellipse becomes a circle then the long axis of the ellipse is just twice the radius of the circle.) If we put this proportionality into equation (38) we obtain the proportionality:

$$(39) \qquad F_r \propto -\frac{m}{r^2}$$

This is in essence Newton's universal law of gravitation saying that the force on the planet is in the direction towards the sun, is proportional to the mass m of the planet, and is inversely proportional to the square of its distance from the sun. The universal law of gravitation is just one step away: If the sun and the planet are treated as two bodies attracting each other, rather than merely the sun attracting the planet, that is, if the sun and the planet stand in a symmetrical relation to each other, then if the force is proportional to the mass of the planet m it must also be proportional to the mass of the sun, which we shall call M. Putting in a constant of proportionality G, called the universal gravitational constant we obtain Newton's law of universal gravitation:

$$(40) \qquad F_r = -G\frac{Mm}{r^2}, \qquad F_\theta = 0$$

Section 5. General Discussion of the Radial Motion of a Planet

By the radial motion of a planet we mean the way in which its distance r from the sun depends on time. We want to be able to get general results about this dependence, without having to involve the way in which the angle θ depends on time. We start by substituting equations (40) and (22) into equation (30) to get

$$(41) \qquad -\frac{GMm}{r^2} = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$$

The first remark about equation (41), and it is a most important remark, is that the mass of the planet, m, cancels out of this equation. This means that the motion of the planet around its centre of attraction is completely independent of the mass of that planet. Thus if the earth had twice the mass that it actually has, it would still move about the sun in exactly the same way. (We are ignoring the small corrections arising from the fact that there are gravitational forces between the earth and other planets in the solar system.) Similarly the motion of an artificial satellite is, to the same approximation which is a very good approximation, independent of the mass of that satellite. A great big heavy satellite put up by the Russians moves in exactly the same way as a much lighter satellite put up by the Americans. All that matters is the initial velocity, both in magnitude and direction, at the point where the satellite separates from the last rocket stage.

From the practical point of view, the trouble with equation (41) is that it involves not only the dependence of r upon time, but also the dependence of θ upon time. In order to get rid of this dependence, we now make use of Kepler's second law in the form of equation (21). We solve equation (21) for the time derivative dθ/dt, and substitute this result into equation (41) to obtain

$$(42) \qquad -\frac{GM}{r^2} = \frac{d^2r}{dt^2} - \frac{L^2}{r^3}$$

The quantity L appearing on the right side of equation (42) is a constant. It is not the same constant for every planet but it is a constant value for any one planet.

To simplify matters further, we multiply both sides of equation (42) by the quantity dr/dt and we put all terms on one side to obtain:

$$\frac{dr}{dt}\cdot\frac{d^2r}{dt^2} + \left(\frac{GM}{r^2} - \frac{L^2}{r^3}\right)\frac{dr}{dt} = 0 \tag{43}$$

It is an easy matter to verify that equation (43) is completely equivalent to:

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} + \frac{L^2}{2r^2}\right] = 0 \tag{44}$$

However, equation (44) implies that the quantity in the square brackets is a constant of the motion, i.e. it does not change with time.

It turns out that for motion of planets this quantity is negative, and we shall therefore denote it by -H. The result is:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} + \frac{L^2}{2r^2} = -H \tag{45}$$

We now solve this equation for the square of dr/dt to get:

$$\left(\frac{dr}{dt}\right)^2 = -2H + \frac{2GM}{r} - \frac{L^2}{r^2} = \frac{2H}{r^2}(r-r_1)(r_2-r)\ . \tag{46}$$

On the right hand side of this equation we have introduced the two roots r_1 and r_2 of the quadratic equation obtained by setting the left hand side equal to 0.

It is apparent that the square of dr/dt, the quantity on the left hand side of equation (46), can never become negative. Since the quantity on the right hand side is a quadratic expression which is positive if and only if r lies between r_1 and r_2, we therefore obtain the important result that the distance r of the planet from the sun can vary only between r_1 as the lower limit and r_2 as an upper limit. These two values are shown in Figure 5 where we have drawn the orbit schematically in such a way that the point closest

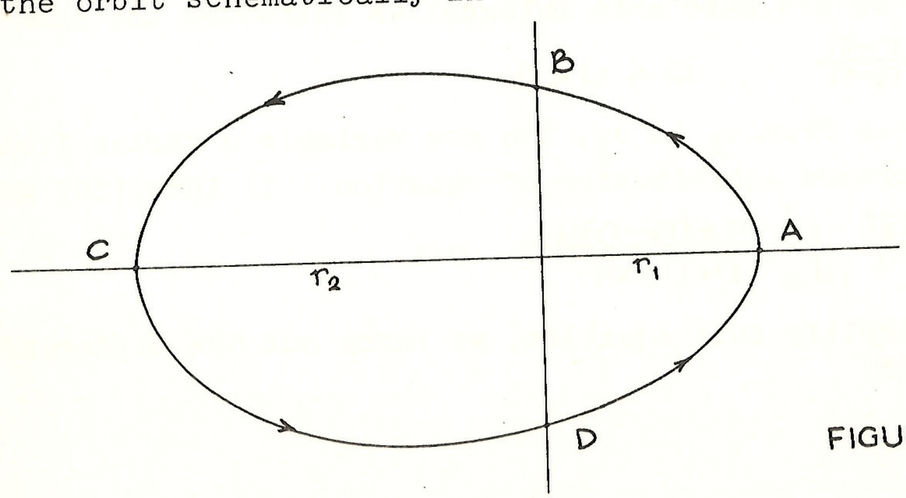

FIGURE 5.

to the sun lies in the positive x direction from the sun. This is not really a restriction, since we can always achieve this by suitably choosing which direction in space we decide to call the x direction. Assuming that the planet revolves around the sun in the counterclockwise direction then the positive square root of equation (46) is appropriate for the value of dr/dt along the section ABC of the orbit, and the negative square root is appropriate for the section CDA. The long axis of the orbit is the sum r_1+r_2. Using the standard relation between the roots of a quadratic equation and the coefficients of that equation, we obtain the relation:

$$(47) \qquad r_1 + r_2 = \frac{GM}{H}$$

We now write down the derivative dr/dt from equation (46) for the path ABC along which the distance r of the planet from the sun is increasing. This gives:

$$(48) \qquad \frac{dr}{dt} = +\sqrt{\frac{2H}{r^2}(r-r_1)(r_2-r)} \qquad \text{Path ABC}$$

Along this path ABC r is a monotically increasing function of the time t. Because of this, an inverse function, t as a function of r, can be defined for this part of the path, and the derivative dt/dr is simply the reciprocal of dr/dt. This yields:

$$(49) \qquad \frac{dt}{dr} = \frac{1}{\sqrt{2H}} \cdot \frac{r}{\sqrt{(r-r_1)(r_2-r)}} \qquad \text{Path ABC}$$

If we now integrate this equation from point A all the way over to point C then we get one-half of the time T that the planet takes to run completely around its orbit and that one-half of the period is given by:

$$(50) \qquad \frac{T}{2} = \frac{1}{\sqrt{2H}} \int_{r_1}^{r_2} \frac{r\cdot dr}{\sqrt{(r-r_1)(r_2-r)}}$$

Equation (50) is an explicit expression, involving a definite integral, for the period of motion of the planet around its orbit.

In order to evaluate this integral we introduce the substitution:

$$(51) \qquad u = \frac{r-r_1}{r_2-r_1} \quad , \quad 0 \leqslant u \leqslant 1$$

As r varies from r_1 to r_2, the new variable u varies from 0 to 1. Straightforward substitution of equation (51) into (50) gives:

$$(52) \qquad T = \sqrt{\frac{2}{H}} \int_0^1 \frac{r_1+(r_2-r_1)u}{\sqrt{u(1-u)}} \, du$$

In order to simplify this equation, we carry out the differentiation indicated below

$$(53) \qquad \frac{d}{du}\sqrt{u(1-u)} = \frac{\frac{1}{2}-u}{\sqrt{u(1-u)}}$$

and we use this to obtain the result:

$$(54) \qquad \int_0^1 \frac{u-\frac{1}{2}}{\sqrt{u(1-u)}} \cdot du = -\sqrt{u(1-u)}\,\Big]_0^1 = 0$$

We therefore write the quantity u in the numerator of the integrand of equation (52) as $(u-\frac{1}{2}) + \frac{1}{2}$ and obtain the result that the $(u-\frac{1}{2})$ part integrates to 0. The remainder is:

$$(55) \qquad T = \frac{r_1+r_2}{\sqrt{2H}} \cdot \int_0^1 \frac{du}{\sqrt{u(1-u)}}$$

The integral in this equation can be evaluated easily by the substitution $w = \sqrt{u}$. This reduces the integral to:

$$(56) \qquad \int_0^1 \frac{du}{\sqrt{u(1\ u)}} = \int_0^1 \frac{2\,dw}{\sqrt{1-w^2}} = 2\sin^{-1} w\,\Big]_0^1 = \pi$$

We now combine equations (47), (55) and (56) to get out final expression for the period of the motion.

$$(57) \qquad T = \frac{\pi}{\sqrt{2GM}}\,(r_1+r_2)^{\frac{3}{2}}$$

This is the precise statement of Kepler's third law of planetary motion. Of the quantities appearing on the right hand side of this equation, π and 2 are absolute mathematical constants, G is the universal gravitational constant, and M is the mass of the sun. Thus the only thing on the right hand side which involves a particular planet at all is the 3/2 power of the long axis of the ellipse r_1+r_2. Thus the period of the motion of a planet is proportional to the 3/2 power of the long axis of the ellipse. This concludes our derivation of the second and third laws of Kepler.

The derivation of the first law is also possible using quite elementary methods but is not possible within the space of this article.

* * *

Follow-up problem:

Show that the path of the planet is a closed curve. (Note: This is a special property of the inverse square law; it is not true, for example, if the force has the form $F_r = \frac{\text{const. } e^{-\alpha r}}{r^2}$

Method: 1. Use equation (21) to write $\frac{d\theta}{dt} = \frac{L}{r^2}$

2. The angle $\Delta\theta$ in going from $r = r_1$ to $r = r_2$ is given by

$$\Delta\theta = \int \frac{d\theta}{dt} \cdot dt = \int \frac{L}{r^2} \cdot dt = \int_{r_1}^{r_2} \frac{L}{r^2} \frac{dr}{\left(\frac{dr}{dt}\right)} .$$

But $\frac{dr}{dt} = \sqrt{2H} \cdot \frac{1}{r} \sqrt{(r-r_1)(r_2-r)}$.

so that $\Delta\theta = \frac{L}{\sqrt{2H}} \int_{r_1}^{r_2} \frac{dr}{r\sqrt{(r-r_1)(r_2-r)}}$

Use the form $\int \frac{dx}{x\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx+2a}{x\sqrt{b^2-4ac}}\right)$

and the identity $2Hr_1r_2 = L^2$ to establish $\Delta\theta = \pi$.

Explain <u>why</u> $\Delta\theta = \pi$ means that the orbit is a closed curve.

* * *

This article was specially written for Parabola by Professor Blatt, of this University's School of Mathematics. It is an expanded version of the talk he gave to a Sydney-wide gathering of First Level Mathematics students at Macquarie University on 5th May 1971 as a demonstration of how First Level Calculus can be applied.

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