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INTERESTING NUMBERS

Mersenne Numbers

A Mersenne number is an integer of the form 2^p-1 where p is a prime. In 1644 Marin Mersenne (1588-1648) asserted that the only values which make 2^p-1 prime were p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 and 257. The list contains some errors; firstly for p = 67 and 257 the numbers are composite. He also omitted p = 61, 89 and 107, although 67 may have been a misprint for 61.

Edouard Lucas (1842-1891), a French mathematician, discovered many of these Mersenne primes and verified that $2^{127}-1$ was prime. The gargantuan $2^{127}-1$ was the largest known Mersenne prime until 1952 when an electronic computer was used to find five larger ones, the largest being $2^{2281}-1$. More recently a computer was used to prove that $2^{3217}-1$ was prime and, even more recently in 1963, Professor Donald R. Gillies of the University of Illinois discovered the largest known Mersenne prime $2^{11213}-1$.

Some Mersenne primes:-

$2^2-1 = 3$
$2^3-1 = 7$
$2^5-1 = 31$
$2^7-1 = 127$
$2^{13}-1 = 8191$
$2^{17}-1 = 131,071$
$2^{19}-1 = 524,287$
$2^{31}-1 = 2,147,483,647$
$2^{61}-1 = 2,305,843,009,213,693,951$

$2^{89}-1$
$2^{107}-1$
$2^{127}-1$
$2^{521}-1$
$2^{607}-1$

$2^{1279}-1$
$2^{2203}-1$
$2^{2281}-1$
$2^{3217}-1$
$2^{11213}-1$

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Perfect Numbers

A positive integer is said to be a perfect number if it is equal to the sum of its divisors including 1 but not itself. Even perfect numbers can be shown to be all of the form $(2^{p-1})(2^p-1)$ where 2^p-1 is a prime. Thus Mersenne primes are always the largest odd factor of even perfect numbers. (See also note by Professor G. Szekeres at end - Ed.)

So far 23 even perfect numbers have been calculated, the smallest being 6 and the largest being $(2^{11,212})(2^{11,213}-1)$. Concerning perfect numbers Mersenne himself wrote "We see clearly how rare are perfect numbers and how right we are to compare them with perfect men."

The first four perfect numbers are:-

$$6 = (2^{2-1})(2^2-1) = (2^1)(2^2-1) = 2.3 = 1+2+3$$
$$28 = 2^2(2^3-1) = 4.7 = 1+2+4+7+14$$
$$496 = 2^4(2^5-1) = 16.31 = 1+2+4+8+16+31+62+124+248$$
$$8128 = 2^6(2^7-1) = 64.127 = 1+2+4+8+16+32+64+127+254+508+1016+2032+4064$$

Other perfect numbers are:-

$$2^{12}(2^{13}-1) = 33,550,336$$
$$2^{16}(2^{17}-1) = 8,589,869,086$$
$$2^{18}(2^{19}-1) = 137,438,691,328$$
$$2^{30}(2^{31}-1) = 2,305,843,008,139,952,128$$

and

$2^{60}(2^{61}-1)$; $2^{88}(2^{89}-1)$; $2^{106}(2^{107}-1)$; $2^{126}(2^{127}-1)$; $2^{520}(2^{521}-1)$; $2^{606}(2^{607}-1)$; $2^{1278}(2^{1279}-1)$; $2^{2202}(2^{2203}-1)$; $2^{2280}(2^{2281}-1)$; $2^{3216}(2^{3217}-1)$; $2^{11212}(2^{11213}-1)$.

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The preceding article was contributed by a school student, Malcolm D. Temperley, of Campbell High School, A.C.T.

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<u>Additional Remarks</u>

As indicated above, it is easy to see that if $q = 2^p-1$ is a Mersenne prime then $n = 2^{p-1}.q$ is a perfect number. Indeed, since q is prime, the proper divisors of n are $1, 2, 2^2, \ldots, 2^{p-1}, q, 2q, \ldots, 2^{p-2}.q$, and their sum is

$$\begin{aligned} 1 + 2 + \ldots + 2^{p-1} &+ q(1 + 2 + \ldots + 2^{p-2}) \\ &= 2^p-1 + q(2^{p-1}-1) \\ &= q.2^{p-1} = n \quad \text{since } q = 2^p-1. \end{aligned}$$

It is a little more difficult to show that every even perfect number must necessarily be of the above form. It is customary to denote by $\sigma(n)$ the sum of all divisors of n, including n itself; then n is perfect if and only if $\sigma(n) = 2n$. Let us write the even perfect number n in the form $n = 2^{k-1}.q$ where $k > 1$ and q is odd. Clearly every divisor of n is obtained by taking the product of a divisor of 2^{k-1} and a divisor of q, and inspection shows that

$$\sigma(n) = (1 + 2 + \ldots + 2^{k-1})\sigma(q) = (2^k-1)\sigma(q).$$

If n is perfect, this must be equal to $2n = 2(2^{k-1}q) = 2^kq$. But $(2^k-1)\sigma(q) = 2^kq$ requires that q be divisible by (2^k-1), since (2^k-1) and 2^k are obviously relatively prime, and so we must have $q = (2^k-1)m$ for some integer m. This gives:

$$(2^k-1)\sigma(q) = 2^kq = 2^k(2^k-1)m \text{ and so } \sigma(q) = 2^km.$$

Now we cannot have $m = q$ as $k > 1$ and so $(2^k-1) > 1$, while $m + q = m + (2^k-1)m = 2^km = \sigma(q)$. But the latter is by definition the sum of q and all other divisors of q; it follows that m is the only other divisor of q hence $m = 1$ and q is prime. But $q = 2^k-1$ hence k must be a Mersenne prime. Furthermore $n = 2^{k-1}q$ and the proof is complete.

No one knows whether there are any odd perfect numbers at all. If they do exist (which is most unlikely) then they must be very large numbers indeed.

G. Szekeres.

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Follow-up problem

Amicable numbers. a and b are said to be amicable if $\sigma(a) = \sigma(b) = a + b$.

Define $P_n = 3.2^n-1$, $Q_n = 9.2^{2n-1}-1$ $(n = 1,2,\ldots)$. Show that if P_{n-1}, P_n and Q_n are prime numbers, then $a = 2^nP_{n-1}.P_n$ and $b = 2^nQ_n$ are amicable.

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DID YOU KNOW ...

Fermat Numbers

Did you know that there was a misprint in our factorisation of F(5)? This should have read F(5) = 641 × 6,700,147.

Did you know that the actual factors of F(7), which is the number 340282366920938463463374607431768211457, are 59649589127497217 and 5704689200685129054721? These were discovered by M.A. Morrison of the University of California and J. Brillhart of the University of Arizona (from the Bulletin of the American Mathematical Society, March 1971).

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SOME QUICKIES

1. Show that any whole number of the form abc, abc where a, b and c are different digits, is divisible by 7, 11 and 13.

2. A car is travelling at x m.p.h. What is the speed of a point on the edge of a wheel at the moment it is on the bottom of the wheel?

(Answers on p. 40)

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