SOLUTIONS TO IBM SCHOOL MATHEMATICS COMPETITION 1971

Junior Division

1. x and y are unequal positive integers. Prove that xy does not divide x^2+y^2.

Answer Let d = (x,y) - the g.c.d. of x and y - so $x = x_1 d$, $y = y_1 d$ with $(x_1, y_1) = 1$. Then $\frac{x^2+y^2}{xy} = \frac{x_1^2+y_1^2}{x_1 y_1}$ and this can only be an integer if $x_1 = 1$ and $y_1 = 1$: contradiction. So xy does not divide x^2+y^2.

2. Without calculating numerical values prove that

$$\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}} < 2\sqrt[3]{3}.$$

Answer This inequality is equivalent to $\sqrt[3]{\frac{1}{3}(3+\sqrt[3]{3})} + \sqrt[3]{\frac{1}{3}(3-\sqrt[3]{3})} < 2$ or to $\sqrt[3]{1+x} + \sqrt[3]{1-x} < 2$ where $x = 3^{-2/3}$.

$$\begin{aligned}\text{Now } (2-\sqrt[3]{1+x})^3 &= 8-(1+x)-4.3\sqrt[3]{1+x}+2.3(\sqrt[3]{1+x})^2\\ &= (1-x)+6[1-2\sqrt[3]{1+x}+(\sqrt[3]{1+x})^2]\\ &= (1-x)+6[\sqrt[3]{1+x}-1]^2\\ &> 1-x.\end{aligned}$$

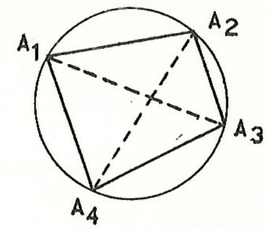
So $2-\sqrt[3]{1+x} > \sqrt[3]{1-x}$ or $\sqrt[3]{1+x}+\sqrt[3]{1-x} < 2$.

3. It is known that among all n-sided polygons inscribed in a circle, the regular n-gon has the greatest perimeter. Using this fact, prove that

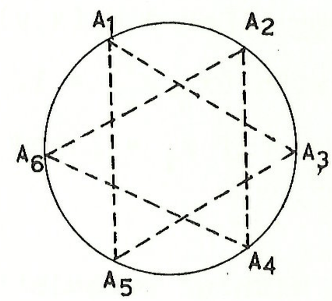
(i) if four points A_1, A_2, A_3, A_4 are chosen on a circle such that the sum of all six distances $A_i A_j$ is maximal then the points are vertices of a square.

(ii) if six points A_1, A_2, A_3, A_4, A_5, A_6 are chosen on a circle such that the sum of all fifteen distances $A_i A_j$ is maximal then the points are vertices of a regular hexagon.

Answer (i) The length of each chord A_1A_3 and A_2A_4 is greatest when it is a diameter. The sum of the distances A_1A_2, A_2A_3, A_3A_4, A_4A_1 is greatest when $A_1A_2A_3A_4$ is a regular 4-gon, i.e. a square. As this ensures that A_1A_3 and A_2A_4 are diameters the statement is proved.



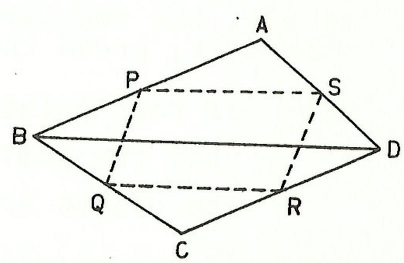
Answer (ii) The sum of the distances = $x + y_1 + y_2 + z$ where x = the perimeter of the hexagon $A_1A_2A_3A_4A_5A_6$, y_1 = the perimeter of the triangle $A_1A_3A_5$, y_2 = the perimeter of the triangle $A_2A_4A_6$, and $z = A_1A_4 + A_2A_5 + A_3A_6$. x is greatest when the hexagon is regular, each of y_1, y_2 is greatest when the respective triangles are regular (i.e. equilateral) while z is greatest when each of the 3 chords is a diameter. As all these conditions are fulfilled simultaneously if the hexagon is regular, we are done.



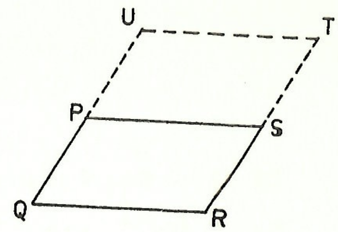
4. (i) Prove that if ABCD is a convex quadrilateral (i.e. every interior angle is less than 180°), the midpoints of its sides are the vertices of a parallelogram whose area is half of the area of ABCD.

(ii) PQRS is a given parallelogram, ABCD a convex quadrilateral such that P is the midpoint of AB, Q the midpoint of BC, R the midpoint of CD, and S the midpoint of DA. Find the region in which A must lie. Give reasons for your answer.

Answer (i) In ΔABD, PS is parallel to BD by a well-known theorem. As also ΔAPS is similar to ΔABD, we get PS = ½BD and the altitude through A to PS is half the altitude through A to BD. Consequently area ΔAPS = ¼ area ΔABD. Similarly in ΔCBD, QR is parallel to BD and area ΔCQR = ¼ area ΔCBD. So QR and PS are equal and parallel whence PQRS is a parallelogram while area ΔAPS + area ΔCQR = ¼ area ABCD. But again area ΔBPQ + area ΔDSR = ¼ area ABCD and thus the area result follows.



Answer (ii) From (i) area $\Delta ASP = \frac{1}{4}$ area ΔABD $< \frac{1}{4}$ area ABCD $= \frac{1}{2}$ area PQRS $=$ area ΔPSQ. (If C is very close to QR, the area of ΔABD is almost equal to the area of ABCD.) Also A must lie between QP and RS produced. So the required region is the interior of the parallelogram PSTU in which ST = SR.



5. Five diplomats A, B, C, D, E representing five nations take part in a peace conference. The confidential documents are kept in a strongroom which has ten separate locks and can be entered only if all ten locks are unlocked. Each diplomat is given a set of six keys opening six of the locks, such that (i) access to the documents is possible only if at least three of the representatives are present, and (ii) any combination of three diplomats can in fact enter the strongroom. Prove that if neither A nor B has a particular key K then all the others must possess it. Hence find a way of distributing the ten keys among A, B, C, D, E so that the conditions (i) and (i) be satisfied.

Answer (i) As each triplet (A,B,C), (A,B,D), (A,B,E) has all the keys, each of C, D, E must have key K.

(ii) As none of the 10 pairs (A,B), (A,C), (A,D), (A,E), (B,C), (B,D), (B,E), (C,D), (C,E), (D,E) has all 10 keys, each pair must be missing one key at least and, by (i), no two pairs can lack the same key. Consequently each pair lacks exactly one of the keys and so each pair has exactly 9 different keys between them. Thus A might hold keys O, P, Q, R, S, T and B hold keys L, M, N, R, S, T for, as each diplomat has 6 keys 3 must be held in common in each pair. The accompanying table gives a possible solution.

		KEYS									
		K	L	M	N	O	P	Q	R	S	T
DIPLOMATS	A					✓	✓	✓	✓	✓	✓
	B		✓	✓	✓				✓	✓	✓
	C	✓		✓	✓		✓	✓	✓		
	D	✓	✓		✓	✓		✓		✓	
	E	✓	✓	✓		✓	✓				✓

Senior Division

1. If $0 < a_i < 1$, $i = 1, 2, \ldots, n$, where $n > 1$, prove
$a_1 + a_2 + \ldots + a_n - a_1 a_2 \ldots a_n < n-1$.

Answer We prove this by induction on n. As $a_1 + a_2 - a_1 a_2 = 1 - (1-a_1)(1-a_2) < 1$ the result holds for $n = 2$. Assume the result for $n = k$ $(k > 1)$, so $a_1 + a_2 + \ldots a_k - a_1 a_2 \ldots a_k < k-1$. Then
$a_1 + a_2 + \ldots + a_k + a_{k+1} - a_1 a_2 \ldots a_k . a_{k+1} =$
$(a_1+a_2+\ldots+a_k-a_1a_2\ldots a_k) + (a_1a_2\ldots a_k-a_1a_2\ldots a_k a_{k+1}+a_{k+1})$. But the first expression $< k-1$ while $a_1a_2\ldots a_k-a_1a_2\ldots a_k a_{k+1}+a_{k+1} =$
$a_1a_2\ldots a_k+a_{k+1}(1 - a_1a_2\ldots a_k) < a_1a_2\ldots a_k + 1(1 - a_1a_2\ldots a_k) = 1$.
Hence if the statement is true for $n = k$ it is true for $n = k+1$. As it is true for $n = 2$, it is therefore true for all $n > 1$.

2. (i) Let n be an integer greater than 1000, 'a' an arbitrarily given decimal digit $(0 \leq a \leq 9)$. Show that there exists a positive integer $x < n$ such that the first three decimal digits of x/n are equal to a, i.e.

$$\frac{x}{n} = 0.aaa\ldots .$$

(ii) Let n be integer, $991 < n < 1000$. Prove that there exists a positive integer $x < n$ such that the first three decimal digits of x/n are equal, i.e.

$$\frac{x}{n} = 0.aaa\ldots \quad \text{for some a.}$$

(iii) If x is a positive integer less than 721, show that the first three digits in the decimal expansion of $x/721$ cannot all be equal to each other.

Answer (i) We show that given digits $0 \leq a, b, c \leq 9$, there is a positive integer $x < n$ such that $\frac{x}{n} = 0.abc\ldots$. For if not there would be a positive integer $y < n-1$ such that $\frac{y}{n} = 0.a_1b_1c_1\ldots$ and $\frac{y+1}{n} = 0.a_2b_2c_2\ldots$ with $0.a_1b_1c_1 < 0.abc < 0.a_2b_2c_2$. But this can be seen to contradict the fact that $\frac{1}{n} < 0.001$.

Answer (ii) There are 999 decimals of the form 0.001, 0.002, ... 0.999. Each of the n-1 decimals given by x/n, $1 \leq x \leq n$, starts with a different one of these 999. If $991 < n < 1000$ this gives at least 991 different sets of first 3 digits and so leaves no more than 8 unused. But there are 9 decimals of the form 0.aaa in the above set.

Answer (iii) Let $1 \leq a \leq 9$.

(a) $x = 80a$. Then $0.aaa - \frac{x}{721} = \frac{111a}{1000} - \frac{80a}{721} = \frac{31a}{721{,}000}$. As $0 < \frac{31a}{721{,}000} < 0.001$, $\frac{x}{721} \neq 0.aaa\ldots$ for any a.

(b) $x = 80a+1$. Then $\frac{x}{721} - 0.aaa(a+1) = \frac{80a+1}{721} - \frac{111a+1}{1000} = \frac{279-31a}{721{,}000}$. But $0 < \frac{279-31a}{721{,}000} < 0.001$ for $a \neq 9$, while for $a = 9$, $x = 721$, so $\frac{x}{721} \neq 0.aaa$ for any a.

(c) $x < 80$. Then by (a) $\frac{x}{721} < 0.111$.

(d) $80a+2 \leq x \leq 80(a+1)-1$. Then $\frac{x}{721} > 0.aa(a+1)$ by (b) while by (a) $\frac{x}{721} < 0.(a+1)(a+1)(a+1)$.

(e) $x > 640$. Then $\frac{x}{721} > 0.888$ by (b) while $\frac{720}{721} < 0.999$ by (a).

3. Given an interval AB on a line, let S be a set of points in CAB with the following properties:

(i) $A \in S$, $B \in S$.

(ii) If $X, Y \in S$ then also $Z \in S$ where Z is between X and Y such that the distance XZ is one tenth of the distance XY.

Prove that any given interval in AB contains at least one point of S.

Answer Definition: the interior of the interval CD is the set of all points in the interval except C and D.

Suppose there is at least one interval HK containing no points of S in its interior. Then there exist intervals containing no points of S in their interiors which also contain HK. (HK itself is one such.) There must exist a largest possible such interval which we will call H_1K_1. But then there must exist a point $X \in S$ such that the distance XH_1 is, say, less than $\frac{1}{50}$ of the distance H_1K_1 (because otherwise H_1K_1 could be made larger). Similarly there is a point $Y \in S$ such that the distance K_1Y is less than $\frac{1}{50}$ of the distance

H_1K_1. (Of course we may have $X = H_1$, $Y = K_1$.) But then there must be a point $Z \in S$ such that $XZ - XH_1 = \frac{1}{10}XY - XH_1 = \frac{1}{10}XH_1 + \frac{1}{10}K_1Y - XH_1 + \frac{1}{10}H_1K_1 > \frac{1}{500}H_1K_1 - \frac{1}{50}H_1K_1 + \frac{1}{10}H_1K_1 > 0$ and $ZY - K_1Y = \frac{9}{10}XY - K_1Y > \frac{9}{10}XY - \frac{1}{50}XY > 0$. This places Z in the interior of H_1K_1: contradiction.

4. Find integers a, b and c such that the following fourth-degree polynomial with integral coefficients can be written as the product of two other polynomials with integral coefficients:

$$x(x - a)(x - b)(x - c) + 1.$$

Answer We exclude the trivial factorisations of the polynomial into ±1 multiplied by a quartic factor. So the factorisation is either $(x+\alpha)(x^3+\beta x^2+\gamma x+\delta)$ or $(x^2+\mu x+\alpha)(x^2+\nu x+\delta)$.

In either case $\alpha\delta = 1$. Hence $\alpha = \delta = \pm 1$ and there are four possibilities: (i) $(x+1)(x^3+\beta x^2+\gamma x+1)$ (ii) $(x-1)(x^3+\beta x^2+\gamma x-1)$ (iii) $(x^2+\mu x+1)(x^2+\nu x+1)$ (iv) $(x^2+\mu x-1)(x^2+\nu x-1)$.

(i) $x(x-a)(x-b)(x-c) + 1 = (x+1)(x^3+\beta x^2+\gamma x+1)$. Put $x = -1$ to get $-1(-1-a)(-1-b)(-1-c) + 1 = 0$ i.e. $(1+a)(1+b)(1+c) = -1$ which has four solutions for a,b,c: (0,0,-2), (0,-2,0), (-2,0,0), (-2,-2,-2).

(ii) A similar treatment yields the solutions (0,0,2), (0,2,0), (2,0,0), (2,2,2). For (iii) and (iv) we note that x^4+1 has no factors so (0,0,0) is not a solution.

(iii) $x(x-a)(x-b)(x-c) + 1 = (x^2+\mu x+1)(x^2+\nu x+1)$. Put $x = a$ to get $1 = (a^2+\mu a+1)(a^2+\nu a+1)$ and the right hand side must equal (1)(1) or (-1)(-1). But putting $x = b$ we get $1 = (b^2+\mu b+1)(b^2+\nu b+1)$ and putting $x = c$ we get $1 = (c^2+\mu c+1)(c^2+\nu b+1)$ so we have

$$(a^2+\mu a+1) = (a^2+\nu a+1)$$
$$(b^2+\mu a+1) = (b^2+\nu a+1)$$
$$(c^2+\mu a+1) = (c^2+\nu a+1)$$

and so $(\mu-\nu)a = 0$, $(\mu-\nu)b = 0$, $(\mu-\nu)c = 0$. As we cannot have $a = 0$, $b = 0$, $c = 0$ simultaneously, $\mu = \nu$.

Then $x(x-a)(x-b)(x-c) = (x^2+\mu x+1)^2 - 1 = x(x+\mu)(x^2+\mu x+2)$.

(Continued)

But $x^2+\mu x+2$ only has factors if $\mu^2 - 8$ is a perfect square which gives $\mu = \pm 3$ and $x(x-a)(x-b)(x-c) = x(x+3)(x+2)(x+1)$ or $x(x-3)(x-2)(x-1)$. This gives 12 solutions: (-3,-2,-1), (-3,-1,-2), (-1,-2,-3), (-1,-3,-2), (-2,-1,-3), (-2,-3,-1), (3,2,1), (3,1,2), (1,2,3), (1,3,2), (2,1,3), (2,3,1).

A similar treatment of (iv) gives $\mu = \pm 1$ and 12 solutions: (-1,-2,1), (1,2,-1) and their permutations.

5. n players compete in a chess tournament in which no two players play more than one game against each other. At a certain stage of the tournament someone observes that for every two competitors who already had a game against each other there is a third player who has not played against either. What is the largest possible number of games played at this stage? Give reasons for your answer.

Answer (i) Suppose one player A has played games against r players B_1, B_2, ... B_r and has not played games against s players C_1, C_2, ... C_s (note that $r+s+1 = n$). There is some player (obviously a C_i) who has played with neither A nor B_1. Similarly each of the players B_i has not yet completed a game against some C_j. Hence the number of games so far not played number at least s (including A) + r (one involving each B_i) = n-1. Since the total number played at the completion of the tournament is $\frac{n(n-1)}{2}$ the maximum number played at this stage cannot exceed $\frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$.

(ii) We show that it is possible that this maximum number of games should have been played by giving an example. Suppose one player P has played no game, but the remaining (n-1) players have all played one another. The given conditions are clearly satisfied (the third player is always P) and the number of games played is $\frac{(n-1)(n-2)}{2}$.

Thus from (i) and (ii) the largest number of games that could have been played is $\frac{(n-1)(n-2)}{2}$.

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