PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class and school. Model solutions and the names of those who send correct solutions by 20 September 1971 will be published in the next issue of Parabola.

Only those students who have not yet commenced their fourth year of secondary education are eligible to submit solutions of problems in the Junior Section. All students may submit solutions of problems in the Open Section.

JUNIOR

J151 (i) Prove that if k is not a prime then neither is 2^k-1.

(ii) Prove that 2^k+1 is never a prime when k is odd.

J152 (i) Prove that among any 10 consecutive integers there is at least one which is divisible by none of the integers 2, 3, 5, or 7. Thus among the integers 24 to 33 inclusive 29 and 31 are not divisible by any of 2, 3, 5, or 7.

(ii) Prove that whatever 10 consecutive integers we choose, there can never be more than four which are not divisible by 2, 3, 5 or 7.

J153 Factorise $x^8 + x^4 + 1$ into quadratic and linear factors.

OPEN

O154 Houses A and B stand on opposite sides of a river whose banks are straight and parallel. It is desired to construct a footbridge across the river perpendicular to the banks. Find where it should be put in order to provide the shortest route from A to B. NOTE: Either A or B may be on the river bank, both may be, neither may be. You are asked to provide a general solution which will cover all possible cases.

O155 A chess king is placed at the south-west corner of a chess-board. (This corner can be thought of as the square with coordinates (1,1)).

(i) It is first allowed to make only one of the following 2 moves each time: (a) to the adjacent square due east, (b) to the adjacent square due north. How many different paths to the square (4,4) are possible with these restrictions?

(ii) It is now permitted to move to the diagonally adjacent square to the north-east as well as the two moves in (i). How many paths to (4,4) are possible under these conditions?

0156 The numbers 1 to 12 are written on a table top in a circle as on the dial of a clock. White markers are placed on the positions 12 and 6 and black markers on positions 3 and 9. An allowable "move" consists of shifting any marker 5 places in a clockwise direction provided its destination is not already occupied. (Thus a possible first move would be to move a black marker from 3 to 8. Of course, if this black marker had been at 4 instead of 3, it could not have been moved first.)

(i) Find the minimum number of moves required to reach a situation where there are white markers on 12 and 1 and black markers on 2 and 3.

(ii) Prove that it is impossible to achieve the situation: white markers on 6 and 8 and black markers on 7 and 9.

0157 In the following long multiplication P always represents one of the primes 2, 3, 5 or 7. Find the solution and show that it is unique.

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  PPP
   PP
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 PPPP
PPPP
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PPPPP
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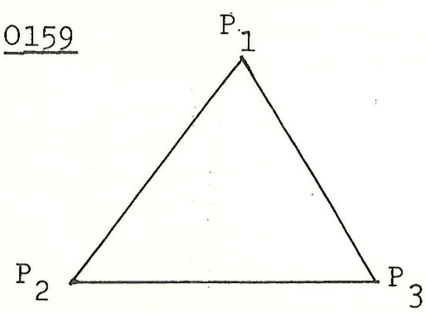
0158 Define $x_1 = \sqrt{6}$, $x_2 = \sqrt{6 + \sqrt{6}}$, $x_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$, ... and, in general, $x_n = \sqrt{6 + x_{n-1}}$. Show that $3 > x_n > x_{n-1}$ for every $n > 2$ but that x_n approaches a limiting value as n becomes large. Find this limiting value.

0159



We take P_4 as the midpoint of P_1P_2, P_5 as the midpoint of P_2P_3, P_6 as the midpoint of P_3P_4 and, generally, P_n as the midpoint of $P_{n-3}P_{n-2}$. P_n approaches a limiting position Q. Find Q and/or give a construction for Q.

0160 The Euler "totient" function $\phi(n)$, where n is any positive integer > 1, is the number of positive integers < n which have no factor in common with n. We define $\phi(1) = 1$.

(i) Prove that $\phi(2) = 1$ is the only odd value of ϕ except $\phi(1) = 1$.

(ii) Observing that $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(6) = 2$ and $\phi(12) = 4$ we see that $\phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(6) + \phi(12) = 12$.

(Continued)

Prove that $\sum_{d|n} \phi(d) = n$ for all $n > 0$, where the left-hand side is the sum of the values of ϕ for each divisor d of n.

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A SPECIAL PROBLEM

Express each of the fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}$ as fractions in which (a) the numerator has 4 digits and the denominator 5 digits and (b) the digits used are 1 to 9, each one being used only once. To get you started $\frac{1}{2}$ is $\frac{6729}{13458}$; note that each digit 1 to 9 appears exactly once.

Contributed by Malcolm Temperley of Campbell High School, ACT.

Additional Question How many solutions are there for each of the 8 fractions $\frac{1}{2} \dots \frac{1}{9}$? (Malcolm provided one solution for each. Is this all? - Ed.)

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WHAT'S WRONG?

A man went to his boss to ask him for a raise. "You have a hide asking me for a raise", his boss replied, "when you do not work". This is the boss's argument:

There are 365 days in a year and the man works 8 hours (or 1/3 day) per day. Thus he effectively works only 1/3 of 365 = 122 days. He has 3 weeks' holiday, leaving 101 days; there are 48 more weekends, leaving 5 days. Finally the man does not work on New Year's Day, Good Friday, Easter Monday, Christmas Day or Boxing Day. Thus he does no work at all!

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A CARD TRICK

Take a pack of 27 cards and ask your friend to choose one without telling you which one. Deal out 3 "hands" each containing a pile of 9 cards and ask your friend which pile his chosen card is in. Pick up the piles with the chosen card in the middle pile. After doing this three times, your friend's card will be exactly 14 cards from the top.

No sleight of hand - it's all mathematical!

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