VERTICES, EDGES AND FACES

This is an account of some elementary aspects of the subject known as "algebraic topology". It investigates the placing of nets on surfaces and Euler characteristics.

A "net" is simply a finite number of nodes (knots) with at least three arcs (pieces of string) leading from each node to other nodes. An example is shown in diagram 1. No two arcs join the same two nodes.

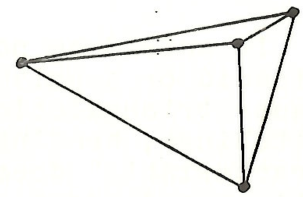

DIAGRAM 1

You perhaps know that if the surface of a sphere is divided up into curvilinear triangles (i.e. triangles with curved sides), then

$$V - E + F = 2$$

where V is the number of vertices of the "triangulation", E is the number of edges, and F is the number of triangular faces. An example is shown in diagram 2, where $V = 4$, $E = 6$, $F = 4$. This formula will be proved and generalised later.

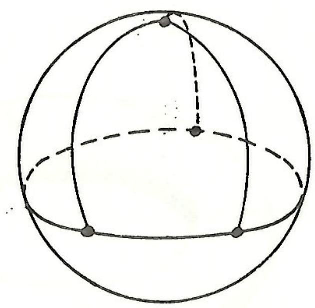

DIAGRAM 2

Suppose that a net with N nodes and A arcs can be placed on the surface of the sphere. It will divide the surface up into (say) M curvilinear polygons, not necessarily triangles. (The example of the net of diagram 1 being placed as in diagram 2 divides the sphere up into four triangles.) However, each curvilinear polygon with $n \geq 4$ sides can readily be subdivided up into n-2 triangles as exemplified by diagram 3 (where $n = 6$). Thus a triangulation can be obtained with

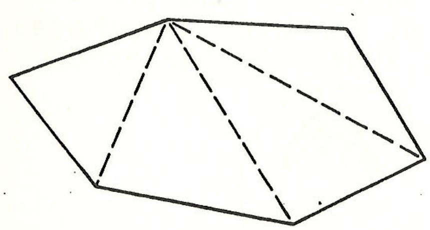

DIAGRAM 3

$V = N$
$E = A + x$ (x is the number of extra lines needed)
$F = M + x$.

So

$2 = V - E + F$
$\quad = N - A + M.$

Thus a condition on the net is obtained that is necessary for its being able to be placed on a sphere. Later examples of nets which do not satisfy this condition will be produced.

Let us return to the formula V - E + F = 2. The same formula holds for an ellipsoid, but not for a torus (the surface of a doughnut). In fact, if a given closed surface is triangulated, V - E + F turns out to be independent of the number and arrangement of the triangles. It is sometimes called the "Euler characteristic" of the surface, after the Swiss mathematician Leonhard Euler (1707-1783). The essence of the proof of the formula is showing this independence of the triangulation. The value of V - E + F can then be calculated by evaluating it for one simple example, such as that of diagram 2.

It should be noted that the proof of this formula works for an ellipsoid or a polyhedron just as well as for a sphere. This is because triangulations are preserved by distortions (squashings, stretchings, bendings and the like). It is clear that the "Euler characteristic" does not depend upon the precise position of the vertices and edges of the triangulation, but only on the general shape. Diagram 4 should help you appreciate that a torus can be distorted into a sphere with a handle.

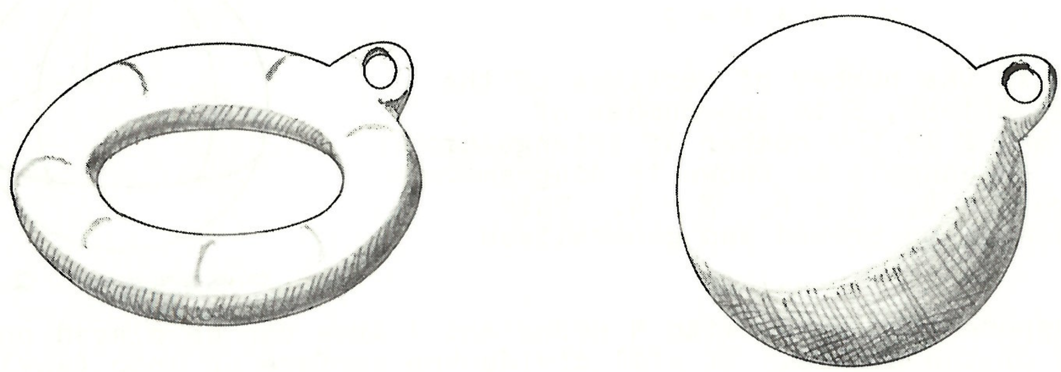

DIAGRAM 4

The simplest, but degenerate, triangulation of the surface of a sphere is obtained by taking three points on it as vertices, three curves (not intersecting elsewhere) as edges and the two triangles so obtained as faces. See diagram 5, in which the shaded area is one triangle, and all the rest of the sphere is the other. Here V - E + F = 3-3+2 = 2, and so the formula holds. Another with four vertices has already been produced. Up to distortions, it is the only triangulation with four vertices.

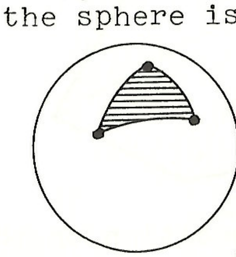

DIAGRAM 5

The proof proceeds by mathematical induction. We assume that the formula has been proved for all triangulations with $k \geq 4$ vertices and prove it for triangulations with k+1 vertices. This will prove the formula in general.

Choose a vertex from a triangulation with k+1 vertices, E edges and F faces. It will have $n \geq 3$ edges emanating from it. Consider first the case $n = 3$. Then the triangulation obtained by removing this vertex and the three edges (as in diagram 6) has k vertices, E-3 edges and F-2 faces. However, by the induction hypothesis, $k - (E-3) + (F-2) = 2$. Hence $(k+1) - E + F = 2$, as required. Secondly, consider the case $n > 3$. In this case, after this vertex and the n edges have been removed, (n-3) edges may be drawn in as in diagram 7. This gives a new triangulation with k vertices, (E-3) edges and (F-2) faces. Again the induction hypothesis shows that $k - (E-3) + (F-2) = 2$. So $(k+1) - E + F = 2$. The proof is now complete.

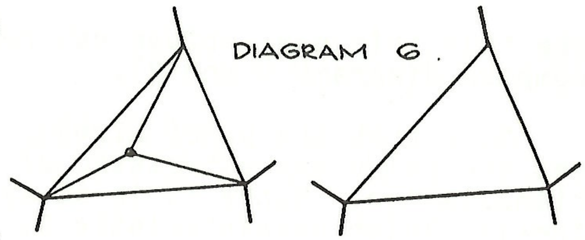

DIAGRAM 6.

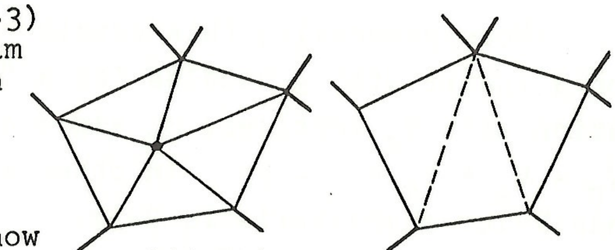

DIAGRAM 7.

It is worth noting that the same formula holds for divisions of the surface into F polygons. This generalisation is obtained by the same method as that used earlier to consider the placing of nets on the sphere.

Now suppose that a sphere is triangulated in such a way that there are two triangles which have no common vertex - as in diagram 8. Imagine a triangular hole constructed through the sphere between the two triangles. The sides of the hole will be three curvilinear rectangles. Draw in one diagonal of each rectangle as in diagram 9.

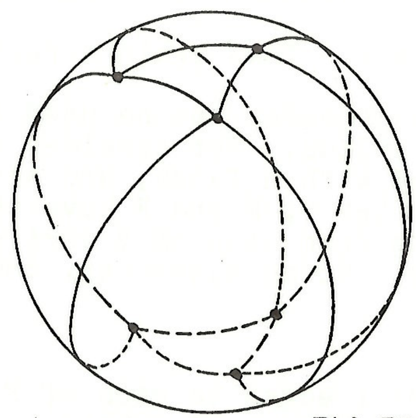

DIAGRAM 8

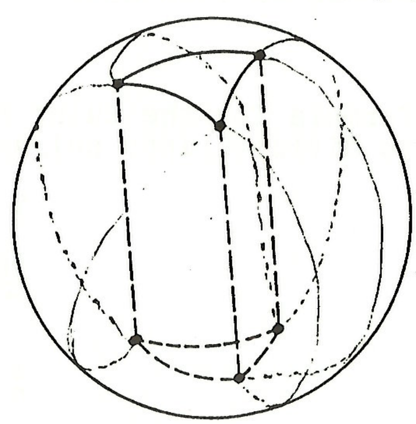

DIAGRAM 9

If V, E and F were the numbers for the original triangulation of the surface of the sphere, and V', E' and F' are the numbers for the triangulation of the surface of the sphere with a hole, $V_1 = V$, $E_1 = E + 6$, $F_1 = F + 4$.

So

$$V_1 - E_1 + F_1 = V - E + F - 2 \\ = 0.$$

Note that this new surface can readily be distorted into a torus (compare diagrams 4 and 9).

Note that the proof already presented applied to a torus shows that the Euler characteristic of a torus is independent of the triangulation. Hence it is always 0. The sphere with a handle has likewise Euler characteristic 0.

More generally, this construction may be repeated to show that on a sphere with g handles (or a "sphere with g holes") a triangulation with V_g vertices, E_g edges and F_g faces is such that $V_g - E_g + F_g = 2 - 2g$. In other words a closed surface of "genus" g has Euler characteristic 2-2g.

Now we may return to examining the possibility of placing a net with N nodes and A arcs on a sphere with g handles. As before, we must have

$$2 - 2g = N - A + M$$

where $M \geq 2$. Thus again a restriction is placed on N and A.

This theorem can be used to show that there are only five regular solids. For consider a regular solid with each face an r-gon and with n faces, and accordingly n edges, meeting at each vertex. Let V, E and F have the usual meaning. There are n edges emanating from each of V vertices, but each edges emanates from two vertices. So an elementary counting argument shows that

$$nV = 2E.$$

There are r edges to each of F faces, but each edge is the edge of two faces. So we also have

$$rF = 2E.$$

But the formula for the Euler characteristic (which, remember, also applies to division into polygonal regions) shows that

$$V - E + F = 2,$$

and so

$$\frac{2E}{n} - E + \frac{2E}{r} = 2,$$

that is

$$\frac{1}{n} - \frac{1}{2} + \frac{1}{r} = \frac{1}{E} > 0.$$

From the meaning of n and r it is clear that each of these numbers must be at least 3. But they cannot both be greater than 3, for this would imply

$$\frac{1}{n} + \frac{1}{r} - \frac{1}{2} \leq \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0.$$

You can likewise show that neither can be greater than 5. Hence the only possible solutions are those in the table below:

n	r	E	V	F	Name
3	3	6	4	4	tetrahedron
3	4	12	8	6	cube
4	3	12	6	8	octahedron
3	5	30	20	12	dodecahedron
5	3	30	12	30	icosahedron

These five possible sets of values do in fact occur. These five regular solids were known to the ancient Greek geometers, and are commonly known as the five Platonic solids.

A generalisation of the above can be made by changing the dimension. If we "triangulate" a point (dimension 0), we find that V = 1, whilst E and F do not occur. So the Euler characteristic of a point is 1. If we "triangulate" a closed curve (dimension 1), faces do not occur, but clearly V = E. So the Euler characteristic of a closed curve is 0. The reader may care to define the Euler characteristic of the "n-dimensional sphere" for n > 2.

Alternatively you may consider the mathematical objects known as graphs. A "graph" is a more general type of net, made up of arcs and nodes subject only to the restrictions that it be connected and that at most one arc joins any two nodes. You may care to find conditions for a graph to be able to be placed on a surface.

Peter Donovan.

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Parabola is indebted to John Bilmon, who prepared the diagrams for this article and for all other articles, with the exception of The Harmonograph. John is a First Year Architecture student at the University of New South Wales.

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