SCHOOL MATHEMATICS COMPETITION 1971: EXAMINERS' COMMENTS

Junior Division

Question 1. Essentially this reduces to proving that $x|y$ (x divides y) and $y|x$ simultaneously implies that $x = y$ which is a contradiction. First prize winner, Robert Kuhn of Sydney Grammar, said: if $k = \text{g.c.d.}(x,y)$ and $x = ka$, $y = kb$, then $xy(x^2+y^2) = a/b + b/a$. As $x \neq y$, one of the fractions, say a/b, > 1, and then $b/a < 1$. So put $a/b = X + d/b$, $d < b$. Then $d/b + b/a = 1$ giving $b^2/a = d-b$, an integer. This means $a|b^2$; contradiction. (We really need to explain why we cannot have $a = 1$.)

Tony Holzkerr of Cabramatta High said: $xy(x^2+y^2) = k$, an integer, gives $x^2-kxy+y^2 = 0$ so the discriminant (b^2-4ac) of the L.H.S. is a perfect square. But this $= k^2-4 = (k-2)(k+2)$; impossible. (This needed proof: $k^2-4 = h^2$ means $(k+h)(k-h) = 4$ with $(k+h) > (k-h)$ unless $h = 0$ and $k = 2$. But this gives $x = y$. Otherwise we would need $k+h = 4$, $k-h = 1$: impossible.)

Essentially correct solutions were given by first prize winner Keith Burns of Campbell High, third prize winner Martin Ellison of Sydney Boys High, Gregory Rose of Fort St. High, Thomas Fowler of Blakehurst High, Douglas McLeod of Beacon Hill High and Peter Monro of Sydney Grammar. Good attempts were made by Allan Pollock of Sydney Grammar and Terry O'Brien of Newcastle Boys High.

Robert Kuhn suggested the generalisation: $x \neq y$ implies xy does not divide (x^n+y^n); Martin Ellison suggested (i) $(x_1x_2...x_n) + (x_1{}^2+x_2{}^2+...+x_n{}^2)$ and (ii) $(x_1x_2...x_n) + (x_1+x_2+...+x_n)^2$. Peter Gill of Sydney Grammar suggested an application to right angled triangles.

Question 2. Martin Ellison came nearest to the solution given in our last issue. Incidentally, far too many competitors thought that $(a+b)^3 = a^3+b^3$.

Question 3. (i) Many proved that the sum of the lengths of sides and diagonals of a square was maximal among quadrilaterals, but did not prove what the question asked, viz: that the square was the only one with this property. (ii) was similarly treated. Timothy Aisbett of Murwillumbah High and Keith Burns suggested generalisation to figures of n sides.

Question 4. Nobody used the area facts from (i) to prove (ii) but good proofs of (ii) were given by Keith Burns, Robert Kuhn. Douglas McLeod, Ian Morris of Sydney Grammar and Iain Johnstone of Canberra Grammar.

Question 5. This was the one done best; most serious competitors got it right. (NOTE: criticisms or justifications of generalisations given above would be welcome, as would further generalisations - Ed.)

Senior Division

The correct solutions written by candidates differed only in details of presentation from those published on pages 22 to 25 last time. The five questions appeared to be of equal difficulty.

Unfortunately "990" was printed in Question 2(ii) instead of "991". Due adjustment was made in the marking were appropriate so that no candidate was disadvantaged.

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Answer to Follow-up Problem to "Interesting Numbers" in Vol 7 No 2

Preliminary Let $N = XY$, where X, Y are integers with no factors in common except 1, and let $\{x_1, x_2, \ldots, x_r\}$ be all the divisors of X, including 1 and X, and $\{y_1, y_2, \ldots, y_3\}$ be all the divisors of Y. Then every divisor of N is of the form xy. Then $\sigma(N)$ = sum of the divisors of $N = (x_1+x_2+\ldots+x_r)(y_1+y_2+\ldots+y_s) = \sigma(X)\sigma(Y)$.

Main answer $P_{n-1} = 3.2^{n-1}-1$, $P_n = 3.2^n-1$, $Q_n = 3^2.2^{2n-1}-1$. All are prime, all are odd for $n > 1$. $\sigma(a) = \sigma(2^n.P_{n-1}.P_n) = \sigma(2^n)\ (P_{n-1}P_n) = \sigma(2^n)\sigma(P_{n-1})\sigma(P_n)$.

$$\sigma(b) = \sigma(2^nQ_n) = \sigma(2^n)\sigma(Q_n)$$

$$\sigma(P_{n-1}) = P_{n-1}+1 = 3.2^{n-1}.\sigma(P_n) = 3.2^n.\sigma(Q_n) = Q_n+1 = 3^2.2^{2n-1}.$$

$$\sigma(2^n) = 1+2+2^2+\ldots+2^n = \frac{2^{n+1}-1}{2-1} = 2^{n+1}-1.$$

Then (i) $\sigma(a) = (2^{n+1}-1)3.2^{n-1}.3.2^n = 3^2.2^{2n-1}.(2^{n+1}-1)$

(ii) $\sigma(b) = (2^{n+1}-1).3^2.2^{2n-1}$

(iii) $a+b = 2^n[P_{n-1}P_n+Q_n].(P_{n-1}P_n = 3^2.2^{2n-1}-3.2^n-3.2^{n-1}+1$ and $2^n+2^{n-1} = 3.2^{n-1}.)$

$$= 2^n(3^2.2^{2n-1} - 3^2 2^{n-1}+3^2.2^{2n-1})$$

$$= 2^{2n-1}.3^2(2^n+2^n-1) = 2^{2n-1}.3^2(2^{n+1}-1).$$