SOLUTIONS

Solutions to Problems 151-160 in Vol 7 No 2

The names of successful problem solvers appear after the solution to Problem 160.

J151 (i) Prove that if k is not a prime then neither is 2^k-1.

(ii) Prove that 2^k+1 is never a prime when k is odd.

Answer (i) If k = pq, where p and q are integers greater than one, then

$$2^k-1 = 2^{pq}-1 = (2^p)^q-1 = b^q-1 \text{ where } b = 2^p$$
$$= (b-1)(b^{q-1} + b^{q-2} + \dots + b + 1).$$

Both factors are integers greater than one.

Answer (ii) $2^k+1 = (2+1)(2^{k-1} - 2^{k-2} + 2^{k-3} - \dots - 2^1 + 1)$ when k is odd.

Hence 3 is a factor.

J152 (i) Prove that among any 10 consecutive integers there is at least one which is divisible by none of the integers 2, 3, 5, or 7. Thus among the integers 24 to 33 inclusive 29 and 31 are not divisible by any of 2, 3, 5, or 7.

(ii) Prove that whatever 10 consecutive integers we choose, there can never be more than four which are not divisible by 2, 3, 5 or 7.

Answer (i) It is very easy to prove that every third odd integer is divisible by 3, every fifth one is divisible by 5 and, in fact, if p is an odd prime, every pth odd integer is divisible by p. Hence of the five consecutive odd integers, at most 2 are divisible by 3, exactly 1 is divisible by 5, and at most 1 is divisible by 7. Hence there must be at least 1 (i.e. 5-2-1-1) which is not divisible by any of these primes.

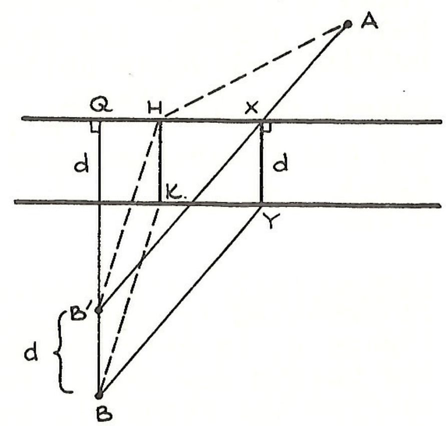
Answer (ii) If the middle odd number in the set is divisible by 3, 5, 7, then none of the four remaining odd numbers have any of these factors. Since of the 10 numbers five are even and one is an odd multiple of 5, there can never be more than four having no factors among 2, 3, 5 and 7.

J153 Factorise $x^8 + x^4 + 1$ into quadratic and linear factors.

Answer
$$\begin{aligned} x^8 + x^4 + 1 &= (x^8 + 2x^4 + 1) - x^4 \\ &= (x^4 + 1)^2 - (x^2)^2 \\ &= (x^4 - x^2 + 1)(x^4 + x^2 + 1) \\ &= [(x^4 + 2x^2 + 1 - 3x^2)][(x^4 + 2x^2 + 1) - x^2] \\ &= [(x^2 + 1)^2 - (\sqrt{3}x)^2][(x^2 + 1)^2 - x^2] \\ &= (x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1) \; . \; (x^2 - x + 1)(x^2 + x + 1). \end{aligned}$$

None of these quadratic factors can be further factorised into real linear factors.

0154 Houses A and B stand on opposite sides of a river whose banks are straight and parallel. It is desired to construct a footbridge across the river perpendicular to the banks. Find where it should be put in order to provide the shortest route from A to B. NOTE: Either A or B may be on the river bank, both may be, neither may be. You are asked to provide a general solution which will cover all possible cases.



Answer Construct a line BQ perpendicular to the river, and a point B' on this line whose distance from B equals the width of the river. Let AB' intersect the bank of the river nearest A at X. Then the bridge, XY, should be built at this point of the river.

Proof Let any other bridge, HK, perpendicular to the banks be constructed. Construct the lines B'H, AH, BK, BY.

BYXB' is a parallelogram since BB' and YX are equal in length and parallel. Hence BY and B'X are equal in length. Similarly BK and B'H are of equal length. Hence AX+XY+BY = AX+XY+B'X = AB'+XY < (AH+B'H)+XY = AH+BK+HK.

0155 A chess king is placed at the south-west corner of a chess-board. (This corner can be thought of as the square with coordinates (1,1)).

(i) It is first allowed to make only one of the following 2 moves each time: (a) to the adjacent square due east, (b) to the adjacent square due north. How many different paths to the square (4,4) are possible with these restrictions?

(ii) It is now permitted to move to the diagonally adjacent square to the north-east as well as the two moves in (i). How many paths to (4,4) are possible under these conditions?

Answer (i) The answer can quickly be obtained by constructing the table shown in figure 1. The number in each square of the board shows the number of paths by which the king can be moved to this square from square x using only king moves due east or due north. First the ones along the bottom row and lefthand column can be entered, the remaining numbers being entered systematically, using the fact that each is the sum of the two numbers due left and directly below. For example, the 20 in square (4,4) is obtained by adding the 10's adjoining it. Why? Simply because at the last move the piece must have moved from one or other of these squares. Hence there are 10 different paths to (4,4) ending with a move north, and another 10 ending with a move east, a total of 20.

1				
1	4	10	20	
1	3	6	10	
1	2	3	4	5
x	1	1	1	1

Another solution is as follows. To proceed from (1,1) to (h,k) a total of (h+k-2) moves must be made, (h-1) due east, and k-1 due north. The problem is the same as the number of ways of putting h-1 white balls and k-1 black balls into a row of h+k-2 boxes, namely ${}^{h+k-2}C_{h-1}$ (i.e. the number of ways of selecting the h-1 boxes for the white balls). To reach (4,4), this gives ${}^{6}C_{3} = \frac{6.5.4}{3.2.1} = 20$ ways.

Answer (ii) The relevant table to be constructed is that appearing in figure 2, in which after the ones on the bottom row and lefthand column are entered the remaining entries are obtained, using the fact that each is the sum of the numbers immediately west, south, and southwest. Since the square (4,4) bears the number 63 there are 63 different paths to this square using the three allowable moves.

1					
1					
1	7	25	63		
1	5	13	25	41	
1	3	5	7	9	
1	1	1	1	1	1

0156 The numbers 1 to 12 are written on a table top in a circle as on the dial of a clock. White markers are placed on the positions 12 and 6 and black markers on positions 3 and 9. An allowable "move" consists of shifting any marker 5 places in a clockwise direction provided its destination is not already occupied. (Thus a possible first move would be to move a black marker from 3 to 8. Of course, if this black marker had been at 4 instead of 3, it could not have been moved first.)

(Continued)

(i) Find the minimum number of moves required to reach a situation where there are white markers on 12 and 1 and black markers on 2 and 3.

(ii) Prove that it is impossible to achieve the situation: white markers on 6 and 8 and black markers on 7 and 9.

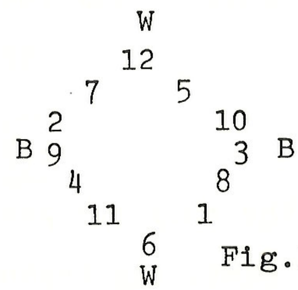


Fig.1

Answer This problem becomes quite simple if we imagine the clockface (or table top) dissected into twelve pieces each bearing one number, and these pieces reassembled, as shown in the accompanying figure. The allowable moves simply consist in moving markers one place clockwise on this diagram (provided there isn't already a marker at this new position). One can answer (ii) immediately by observing that in this new statement of the problem (i.e. on figure 1) one marker can never pass another. Since the white markers alternate with black markers on figure 1 at the commencement of operations, one can never reach a position in which this is not true, such as that with white markers on 6 and 8 and black on 7 and 9.

Part (i) is also simple. The white marker on 12 requires five moves to reach 1, that on 6 requires 6 moves to reach 12, the black marker on 9 (which has to stay "in front" of the white marker on 6 must be moved round to 3 (another 6 moves) and the black marker initially on 3 is similarly pushed round in front of the other white marker and requires 7 moves to reach the position 2. This gives a total of 24 moves which obviously cannot be improved on.

0157 In the following long multiplication P always represents one of the primes 2, 3, 5 or 7. Find the solution and show that it is unique.

$$\begin{array}{r} PPP \\ PP \\ \hline PPPP \\ PPPP \\ \hline PPPP \\ \hline \end{array}$$

Answer Let us first find all solutions of

$$\begin{array}{r} pp_4p_1 \\ p_2 \\ \hline ppp_5p_3 \\ \hline \end{array}$$

It is obvious that none of p_1, p_2 and p_3 are 2, and that p_3 and either p_1 or p_2 must be 5. Try $p_2 = 5$. If p_1 is 3, one has to

(Continued)

"carry" 1 and no allowable choice of p_4 will give a prime for p_5. Similarly if p_1 is 7, the only way to get p_5 prime is by taking $p_4 = 2$, and then we carry 1 to the last stage of multiplication, and this breaks down as before. Hence p_1 must be 5.

Proceeding in this same fashion one finds that if $p_1 = 5$, the only possibility for p_4 is also 5 and

$$\begin{array}{r} 555 \\ 5 \\ \hline 2775 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 755 \\ 5 \\ \hline 3775 \\ \hline \end{array}$$

are the only two possibilities.

Similarly trying $p_2 = 3$, the only success is

$$\begin{array}{r} 775 \\ 3 \\ \hline 2325 \\ \hline \end{array}$$

and if $p_2 = 7$ we must have

$$\begin{array}{r} 325. \\ 7 \\ \hline 2575 \\ \hline \end{array}$$

Since all the three digit numbers are different there are only four possibilities to test viz 555×55; 755×55; 775×33; and 325×77. Only 775×33 = 25,575 gives an answer containing only prime digits.

0158 Define $x = \sqrt{6}$, $x = \sqrt{6 + \sqrt{6}}$, $x = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$, ... and, in general, $x_n = \sqrt{6 + x_{n-1}}$. Show that $3 > x_n > x_{n-1}$ for every $n > 2$ but that x_n approaches a limiting value as n becomes large. Find this limiting value.

Answer Assuming that x_n approaches a limiting value, $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}$ ... it is not difficult to find the value of x. In fact $x = \sqrt{6 + x}$, whence $x^2 - x - 6 = 0$, $(x-3)(x+2) = 0$. Since all terms x_n are positive (the sign $\sqrt{k}$ always stands for the positive square root of k; you must write $-\sqrt{k}$ for the negative square root) the value of x_n cannot possibly be very near -2. Hence if a limit exists its value is $x = 3$.

Let us set $3 - x_n = d_n$. Thus $d_1 = 3 - \sqrt{6} < .6$ and so on. Since

$$x_n = \sqrt{6 + x_{n-1}}$$

$$3 - d_n = \sqrt{9 - d_{n-1}}$$

(Continued)

$$\text{and } d_n = 3 - \sqrt{9 - d_{n-1}}$$
$$= \frac{(3 - \sqrt{9 - d_{n-1}})(3 + \sqrt{9 - d_{n-1}})}{3 + \sqrt{9 - d_{n-1}}}$$
$$= \frac{d_{n-1}}{3 + \sqrt{9 - d_{n-1}}}.$$

This shows that if d_{n-1} is positive so is d_n and $d_n < \frac{d_{n-1}}{3}$ ... (i)

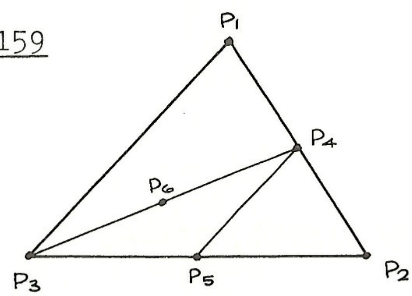
[$0 < d_n < d_{n-1}$ is equivalent to $3 > 3-d_n > 3-d_{n-1}$ or $3 > x_n > x_{n-1}$].

Repeated use of the inequality (i) (which has been proved for all subscripts $n \geq 2$) gives

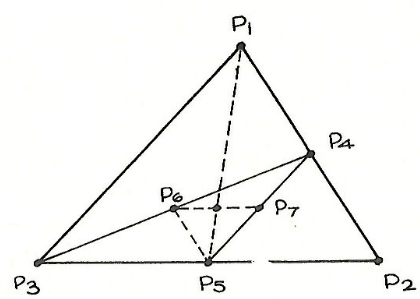
$$d_n < \frac{d_{n-1}}{3} < \frac{1}{3}\left(\frac{d_{n-1}}{3}\right) < \ldots < \frac{d_1}{3^{n-1}} < \frac{.6}{3^{n-1}}$$

which is clearly very small when n is large. Hence x_n ($= 3-d_n$) is very close to 3 for all large n, and the numbers x_n do indeed approach the limiting value 3.

0159



We take P_4 as the midpoint of P_1P_2, P_5 as the midpoint of P_2P_3, P_6 as the midpoint of P_3P_4 and, generally P_n as the midpoint of $P_{n-3}P_{n-2}$. P_n approaches a limiting position Q. Find Q and/or give a construction for Q.



Answer Since the sides of $\Delta P_5P_6P_7$ are parallel to the sides of $\Delta P_1P_2P_3$ it is easy to see that the line P_1P_5 bisects P_6P_7, i.e. cuts it at the point P_9. Hence P_1, P_5, P_9, P_{13}, P_{17}, ... P_{4n+1}, ... all lie on this line, and the limit point must also lie on it. Similarly the limit point lies on P_2P_6. Hence it is at the intersection of these two lines P_1P_5 and P_2P_6. (Can you show that the limit point divides P_1P_5 in the ratio 4:1?)

0160 The Euler "totient" function $\phi(n)$, where n is any positive integer > 1, is the number of positive integers < n which have no factor in common with n. We define $\phi(1) = 1$.

(Continued)

(i) Prove that $\phi(2) = 1$ is the only odd value of ϕ except $\phi(1) = 1$.

(ii) Observing that $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(6) = 2$ and $\phi(12) = 4$ we see that $\phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(6) + \phi(12) = 12$. Prove that $\sum_{d|n} \phi(d) = n$ for all $n > 0$, where the lefthand side is the sum of the values of ϕ for each divisor d of n.

Answer (i) If $k < n$ then k and n are relatively prime if and only if $(n-k)$ and n are relatively prime. It follows that there are the same number of integers relatively prime to n less than $\frac{1}{2}n$ as there are greater than $\frac{1}{2}n$. For even $n > 2$, the integer $\frac{1}{2}n$ is not itself relatively prime to n. Hence $\phi(n)$ is even for $n > 2$.

Answer (ii) List the n fractions $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}$ ... (i).

Express these in lowest terms (e.g. $\frac{n}{n} = \frac{1}{1}$). A fraction k/n cancels down to K/d where $k = Kh$, $n = dh$, and K is relatively prime to d. There are $\phi(d)$ integers K less than d, relatively prime to d, and correspondingly $\phi(d)$ of the fractions (i) cancel down to a form with denominator d for each of the factors d of n. Hence

$$\sum_{d|n} \phi(d) = n.$$

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Solvers of Problems 151-160

Robert Kuhn, Sydney Grammar - 152, 153 (partly right), 154-6, 159 (partly right).
Brian Martin, Cootamundra High - 154-5,157.
Doug McLeod, Beacon Hill High - 154,6,158-9.
Gregory Crannoy, Shore -154-5,157.
Mark Pitt, Bowral High - 154,155 Pt (i),156 Pt (i).
N. Wormald, Sydney Tech.High - 154-160.
Malcolm Temperley, Campbell High ACT - 155 Pt (i), 157,159.
Stephen Hing, St Joseph's - 154-158.
Gabriel Holmik,de la Salle, Kingsgrove - 154-5,157,159.
"H de F" (Would he/she please write in stating name and school! - Ed.) - 154,5, 156 Pt (i), 157-9.

The solutions sent in by N. Wormwald and Doug McLeod were outstanding. Congratulations! Excellent solutions were also submitted by Robert Kuhn - C.D. Cox.

Names of late solvers will be published in Vol 8 No 1.

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READERS' LETTERS

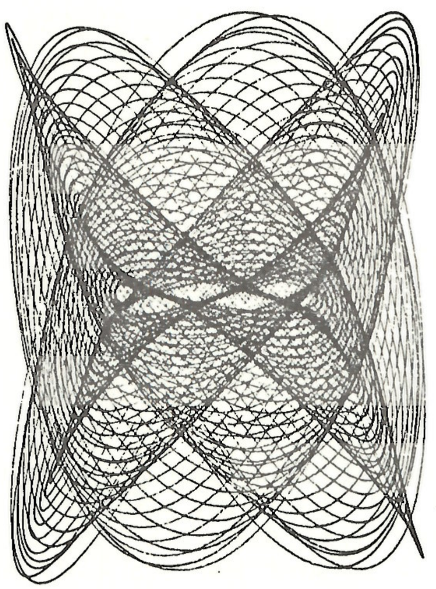
Dear Sir,

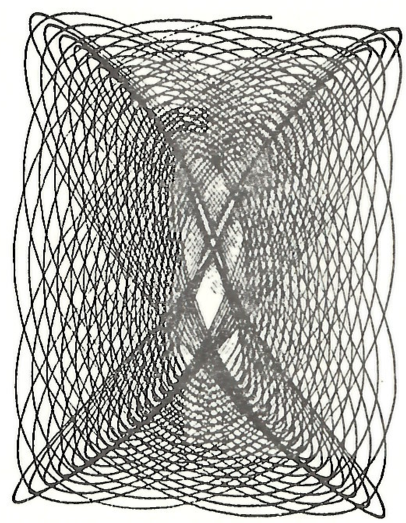
In reference to the article "A Card Trick" in the July/August issue of Parabola this year, the trick is easier to perform if you pick up the piles with the chosen card in the top pile (or bottom pile). Then you may use any number of piles as long as the number of cards used is a multiple of the number of piles e.g. 4 piles and 52 cards. The card your friend chooses will then be the top (or bottom) card (whichever you used) as long as you deal out the piles y times when you use x piles and x^y cards or less.

Yours sincerely,
Michael Young
Caringbah High, NSW.

This is true, but has the disadvantage that the friend can see it working - which is obviously undesirable for a card trick - R.James.

* * *





ANSWERS

Translation to "Our Christmas Message"

"Compliments of the season to the reader". (Get it?)

Answers to Puzzlers

"It's wrong!" - We must have E = 1. From the tens column we must have H = 9 or 0. If H = 9, then N < E and so N = 0, giving T = 9 = H. If H = 0, then (in the thousands column) I = 0 or 1 which is impossible.

"Whodunit?" - Clyde was the guilty man.

"I pass!" - 60 mph.

"An age-old problem" - Bill is the oldest and Arthur the youngest.

"Who's a square?" - A square must end in 1, 4, 5, 6 or 9 and so the second and second last ones are not squares. If a square is divided by 9, its remainder is 0, 1, 4 or 7 and this rules out all but the last (add up the digits!). Thus the last one is a square (it is the square of 50,123,246 if you care to check it!).

"Black out!" - 645 × 721.

Explanation of the card trick

The difference between any number n and the sum of its digits is always a multiple of 9 (why?). If $n < 20$ the difference is 10; $20 \leq n < 30$ the difference must be 18. You could extend this trick by taking $n < 20$ or $n \geq 30$ cards in one of the piles, perhaps.

Answer to Age Problems

Clue 1 The product of their ages equals 36. So combination of factors (3) of 36:

36	1	1	9	2	2
18	2	1	6	6	1
12	3	1	6	3	2
9	4	1	3	4	3.

Clue 2 The sum of their ages equals number of house next door i.e. 38 or 21 or 16 or 14, or 13 or 13 or 11 or 10. Now the census-taker knew the number of the house next door so that his need for yet another clue shows that this number was 13 i.e. ages must be 6, 6, 1 or 9, 2,2.

Clue 3 Eldest plays the piano. Therefore ages are 9, 2, 2 as 6, 6, 1 has no eldest.