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A TRIANGLE SIMILAR TO THE PASCAL TRIANGLE

In Vol 7 No 2 of Parabola, the following problem was set:

A chess king is placed at the south-west corner of a chessboard. (This corner can be thought of as the square with coordinates (1,1)).

(i) It is first allowed to make only one of the following two moves each time: (a) to the adjacent square due east, (b) to the adjacent square due north. How many different paths to the square (4,4) are possible with these restrictions?

(ii) It is now permitted to move to the diagonally adjacent square to the north-east as well as the two moves in (i). How many paths to (4,4) are possible under these conditions?

The problem is solved by constructing a table in which the number of ways of reaching each square is written in the square. The resulting tables for the two parts of the problem are shown below; they are turned so that the square (1,1) is at the top.

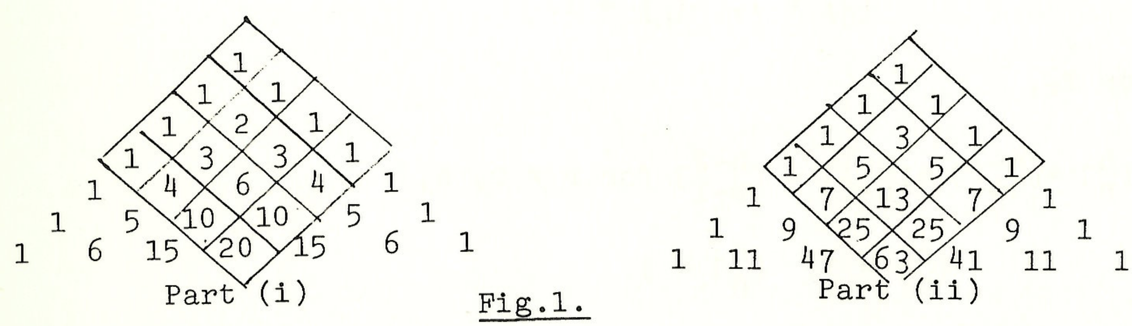


Fig.1.

Both tables may be extended downwards indefinitely to form two triangles. Each horizontal row of the triangle is a diagonal row of the square chessboard.

It follows from the conditions of the problem that in part (i) each number in the triangle is the sum of the two numbers diagonally above it. Thus the triangle for part (i) is the Pascal Triangle.

In part (ii) each number is the sum of the three numbers above it, i.e. in any square of four numbers, the number in the bottom corner is the sum of the other three.

As the rule for forming the triangle in part (ii) is so similar to that for forming the Pascal Triangle, it is interesting to see whether this new triangle possesses properties similar to the many properties of the Pascal Triangle.

First let us define the symbols which will be used in this article. For convenience, the Pascal Triangle will be called T_1, and the other triangle T_2. The rows of both triangles will be numbered from the top downwards, the top row being the zeroth row. The numbers in a row will be numbered from left to right, the number on the extreme left being the zeroth number. The symbol $\binom{n}{r}$ will represent r'th number of the n'th row of T_1; $\left[{n \atop r}\right]$ will represent the r'th number of the n'th row of T_2. (Note that $r \leq n$).

(1) We can now use these symbols to state the defining properties of the two triangles described earlier.

For T_1,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \text{ for } r \neq 0, n,$$

$$\binom{n}{0} = 1, \quad \binom{n}{n} = 1.$$

For T_2,

$$\left[{n \atop r}\right] = \left[{n-1 \atop r}\right] + \left[{n-1 \atop r-1}\right] + \left[{n-2 \atop r-1}\right] \text{ for } r \neq 0, n,$$

$$\left[{n \atop 0}\right] = 1, \quad \left[{n \atop n}\right] = 1.$$

An interesting point about T_2 is that every number in it is odd. This is because every number is either defined to be 1, or is the sum of three odd numbers.

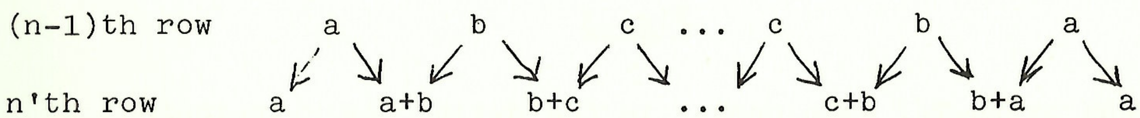
(2) Both triangles are symmetrical about a vertical axis. This is easily proved for T_1, using the fact that if one row is symmetrical, the next row must be too; and for T_2 by noting that if two consecutive rows are symmetrical, the next row will be symmetrical.

(3) Another easily proved property of T_1 is that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} \ldots + (-1)^n \binom{n}{n} = 0,$$

(i.e. the sum of the set of odd-numbered numbers in a row is equal to the sum of the set of even-numbered ones).

It follows from the defining property of T_1 that each number in the (n-1)th row contributes equally to the two consecutive numbers below it in the n'th row. (See Fig. 2.)



'b' contributes equally to a+b and b+c.

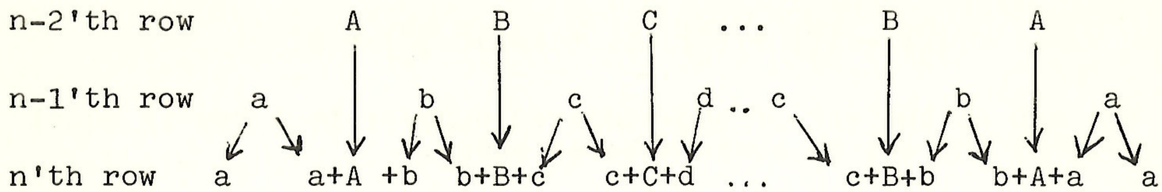
Fig. 2

One of these numbers must be even-numbered, the other odd-numbered.

As this is true for every number in the (n-1)th row, the sum of the odd-numbered numbers must be equal to the sum of the even-numbered numbers, both of these being equal to the sum of the (n-1)th row. (It follows from this that the sum of the n'th row is twice the sum of the (n-1)th row.)

The corresponding property for T_2 is more complicated. For the odd-numbered rows, the property is the same as for T_1, but for the even-numbered rows, the sum of the set containing the middle number is one greater than the sum of the other set.

The property obviously holds for rows 0, 1 and 2. In the general case, it follows from the defining property of T_2 that each number in the (n-2)th row contributes to the number directly below it in the n'th row, while each number in the (n-1)th row contributes equally to the two numbers diagonally below it in the n'th row. (See Fig. 3.)



A contributes to a+A+b; B to b+B+c

'b' contributes equally to a+A+b and b+B+c.

Fig. 3

Thus as in T_1, each number in the n-1'th row contributes equally to the sums of the odd- and the even-numbered numbers in the n'th row. However, each number in the (n-2)th row contributes to only one set of numbers. If two numbers in the (n-2)th row are in the same set, the numbers below them in the n'th row will be in the same set also. (Thus in Fig. 3, A and C are in the same set, as are a+B+b and c+C+d.) It follows from this that if n-2 is odd and thus the sum of the "odd" numbers equals the sum of the "even" numbers, the same will be true in the n'th row; if n-2 is even, the sum of the set containing the middle number of the n'th row will be one greater than the sum of the other set, as in the n-1'th row. Hence, by the Principle of Mathematical Induction, the results of T_2 described earlier in this section are true.

(4) It follows from the defining property of T_1 that

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}.$$

and from this that, for $n > r$,

$$\binom{n}{r} = \binom{n+1}{r+1} - \binom{n}{r+1}. \qquad (*)$$

Using (*) it is possible to show that

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} \dots + \binom{n-1}{r} + \binom{n}{r} = \binom{n+1}{r+1}, \text{ for } n > r.$$

For, from (*), this equals

$$\binom{r}{r} + \binom{r+2}{r+1} - \binom{r+1}{r+1} + \binom{r+3}{r+1} - \binom{r+2}{r+1} \dots + \binom{n}{r+1} - \binom{n-1}{r+1} + \binom{n+1}{r+1} - \binom{n}{r+1},$$

which is equal to $\binom{n+1}{r+1}$ because every other term cancels out, as $\binom{r}{r} = \binom{r+1}{r+1} = 1$.

Similarly it can be proved that

$$\binom{n}{r+1} = \binom{n+1}{r+1} - \binom{n}{r},$$

and hence that

$$\binom{n-r}{0} + \binom{n-r+1}{1} + \dots + \binom{n}{r+1} = \binom{n+1}{r+1}.$$

Fig. 4 shows these results illustrated on the Pascal Triangle. Thus 1 + 2 + 3 + 4 + 5 = 15 = 1 + 4 + 10. A corresponding result is true for any other pair of diagonals similar to those in the diagram.

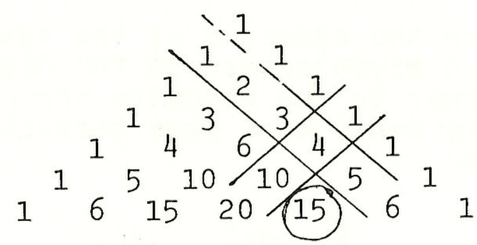


Fig. 4.

$$\genfrac{[}{]}{0pt}{}{n+1}{r+1} = \genfrac{[}{]}{0pt}{}{n}{r} + \genfrac{[}{]}{0pt}{}{n-1}{r} + \genfrac{[}{]}{0pt}{}{n}{r+1} \text{ for } n > r,$$

which gives

$$\genfrac{[}{]}{0pt}{}{n}{r} + \genfrac{[}{]}{0pt}{}{n-1}{r} = \genfrac{[}{]}{0pt}{}{n+1}{r+1} - \genfrac{[}{]}{0pt}{}{n}{r+1} \text{ for } n > r.$$

From this it can be proved that

$$2\genfrac{[}{]}{0pt}{}{r}{r} + 2\genfrac{[}{]}{0pt}{}{r+1}{r} + \dots 2\genfrac{[}{]}{0pt}{}{n-1}{r} + \genfrac{[}{]}{0pt}{}{n}{r} = \genfrac{[}{]}{0pt}{}{n+1}{r+1}$$

for $n > r$. Again, the relation

$$\genfrac{[}{]}{0pt}{}{n-1}{r} + \genfrac{[}{]}{0pt}{}{n}{r+1} = \genfrac{[}{]}{0pt}{}{n+1}{r+1} - \genfrac{[}{]}{0pt}{}{n}{r}$$

gives

$$2\genfrac{[}{]}{0pt}{}{n-r}{0} + 2\genfrac{[}{]}{0pt}{}{n-r+1}{1} + \dots + 2\genfrac{[}{]}{0pt}{}{n-1}{r} + \genfrac{[}{]}{0pt}{}{n}{r+1} = \genfrac{[}{]}{0pt}{}{n+1}{r+1}.$$

The proofs of these results are left to the reader; they are very similar to the proofs for T_1.

(5) The next result is interesting because it is most easily obtained by using the problem at the beginning of the article.

For T_1, the result is that

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}.$$

Consider an (n+1) × (n+1) square. There are $\binom{2n}{n}$ ways of reaching the corner of the board. The n'th row will be the diagonal of the square and this gives another way of working out the number of ways of reaching the corner of the board.

It follows from the symmetry of the square that if there are q ways of reaching a given square on the diagonal, there are q^2 routes from one corner to the opposite corner passing through the given square. Every route must pass through a square on the diagonal. Therefore

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}.$$

The result for T_2 is similar. As for T_1, there are $\sum_{r=0}^{n} \left[{n \atop r}\right]^2$ routes which pass through squares in the n'th row. But there are some routes which pass directly from the (n-1)th to the (n+1)th row. Because of the symmetry of the square, there are $\sum_{r=0}^{n-1} \left[{n \atop r}\right]^2$ of these. It is easy to see from the defining property of T_2 that these are the only possible routes. Therefore

$$\sum_{r=0}^{n} \left[{n \atop r}\right]^2 + \sum_{r=0}^{n-1} \left[{n-1 \atop r}\right]^2 = \left[{2n \atop n}\right].$$

The last two sections involve some more difficult mathematics. In the next section we find the sum of the numbers in the n'th rows of the two triangles; in the final section we consider the formulae which express $\binom{n}{r}$ and $\left[{n \atop r}\right]$ in terms of n and r.

(6) Let us denote by u_n the sum of the numbers in the n'th row of T_1. Then, as was shown in section (3), $u_n = 2u_{n-1}$. Since $u_0 = 1 = 2^0$, $u_n = 2^n$.

Let us call the corresponding number for T_2 v_n. It follows from what was said in section (3) (in particular see Fig. 3) that, for $n \geq 2$,

$$v_n = 2v_{n-1} + v_{n-2}, \text{ with } v_0 = 1, \quad v_1 = 2.$$

This linear difference equation can be solved by standard methods to show that

$$v_n = [(\sqrt{2}+1)^{n+1} + (-1)^n (\sqrt{2}-1)^{n+1}]/(2\sqrt{2})$$

(7) The very well-known Binomial Theorem expresses $\binom{n}{r}$ in terms of n and r, thus

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Unfortunately a similar formula has yet to be found for $\left[{n \atop r}\right]$.

A start can be made by finding formulae for $\left[{n \atop 0}\right]$, $\left[{n \atop 1}\right]$, $\left[{n \atop 2}\right]$ ... for general n. Hopefully these will form a pattern from which a general formula for $\left[{n \atop r}\right]$ can be deduced. It is obvious that $\left[{n \atop 0}\right] = 1$ and that $\left[{n \atop 1}\right] = 2n-1$. It is possible to find $\left[{n \atop 2}\right]$ by the following method.

It follows from the defining property of T that

$$\left[{n \atop r}\right] - \left[{n-1 \atop r}\right] = \left[{n-1 \atop r-1}\right] + \left[{n-2 \atop r-1}\right].$$

Thus $$\left[{n \atop 2}\right] - \left[{n-1 \atop 2}\right] = \left[{n-1 \atop 1}\right] + \left[{n-2 \atop 1}\right]$$

and $$\left[{n-1 \atop 2}\right] - \left[{n-2 \atop 2}\right] = \left[{n-2 \atop 1}\right] + \left[{n-3 \atop 1}\right]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

and finally $$\left[{3 \atop 2}\right] - \left[{2 \atop 2}\right] = \left[{2 \atop 1}\right] + \left[{1 \atop 1}\right].$$

Adding these equations together gives

$$\left[{n \atop 2}\right] - \left[{2 \atop 2}\right] = \sum_{x=2}^{n-1} \left[{x \atop 1}\right] + \sum_{x=1}^{n-2} \left[{x \atop 1}\right]$$

so that

$$\left[{n \atop 2}\right] - 1 = \sum_{x=2}^{n-1} (2x-1) + \sum_{x=1}^{n-2} (2x-1).$$

This can be solved by standard methods to give

$$\left[{n \atop 2}\right] = 2(n-1)(n-2) + 1.$$

The same method can be used to find $\left[{n \atop 3}\right]$, $\left[{n \atop 4}\right]$ ...; in particular,

$$\left[{n \atop 3}\right] = \frac{4}{3}(n-4)(n-3)(n-2) + 2(n-3)(n-2) + 2(n-3) + 1.$$

Readers might like to try to discover the general formula for $\left[{n \atop r}\right]$ using this or other methods.

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The author is joint First Prizewinner in the 1971 School Mathematics Competition - Ed.

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Constructing $\sqrt[3]{2}$ geometrically

You can't, at any rate not with a pair of compasses and unmarked ruler. Many tried; it took over 2,000 years to prove the task impossible. There is however a simple way of doing it if one does a bit of cheating!

Draw the equilateral triangle ABC of sides = 1 unit. Produce AB to D so BD = 1 unit. Join DC. Produce DC and BC as far as necessary to enable you to arrange a ruler or straight piece of paper or card so that it goes through A, meets DC and BC produced at E and F, so that the distance EF is exactly 1 unit. (You would need to mark this 1 unit distance on the straight edge first.) Then AE is the required distance = $\sqrt[3]{2}$.

When you've done the construction, turn to Problem 0178 in this issue on page 32.

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