THE 4 COLOUR MAP PROBLEM

Robert Kuhn of Sydney Grammar has sent in the following attempted proof of this famous undecided conjecture (to wit, that every "map" drawn in an atlas, or on the surface of a globe, can be coloured so that neighbouring countries never receive the same colour provided 4 different colours are available i.e. no map requires the use of a fifth colour). He is sure there must be some error in his argument, and would like it pointed out.

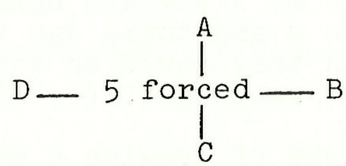
Here is his letter and our comments follow it.

The 5 colour map theorem

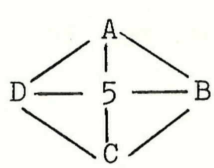
Let us assume that there is a map in which 5 colours are necessary i.e. that a map can be drawn where 4 necessarily different colours are in contact with a fifth.

Let 5 stand for the fifth colour and A, B, C and D for the first four. Lines joining numbers represent contact between areas, no two lines can cross.

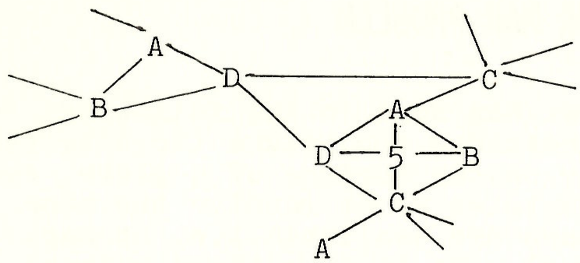
Clearly, to force the 5th colour, there must be an area in contact with A, B, C and D, i.e.



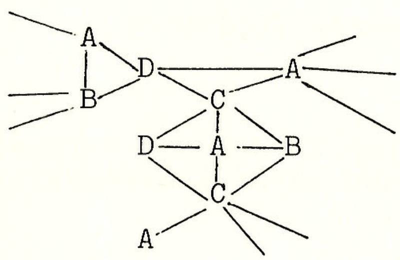
No other areas need be in contact. Therefore if such a map can be drawn, it can also be drawn with contact between adjacent colours i.e.



Consider the pair A+C (or D+B). For them to be different colours, there must be some way of differentiating two areas, and the only method is contact. Therefore there must be a chain of alternate A's + C's (or B's + D's) joining the two areas, because if there were no chain of contact, all the A's could become C's (B's to D's) and vice versa, on one side of the other colour, without any two areas of the same colour becoming in contact. This would mean either a pair of A's (B's) or a pair of C's (D's) i.e.



could become



and the 5th colour would become only one of the four.

But if there is a chain of A's + C's between the original A and C, the plane is divided into 2 and there can be no connecting chain between to D and B, but then they could be the same i.e. B and B (D and D) and the 5th colour would be D (or B).

Therefore there is no way of drawing a map necessitating a 5th colour - QED.

Robert Kuhn
Sydney Grammar School.

(Robert is joint First Prizewinner in the 1971 School Mathematics Competition - Ed.)

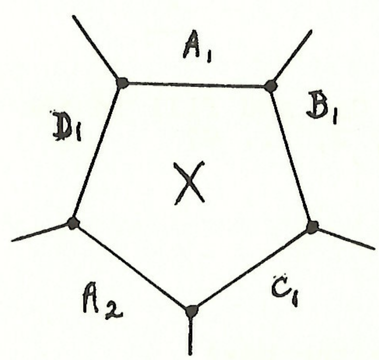
Comments

Well, did you find a fallacy?

The error is associated with the paragraph near the beginning "No other areas ... contact between adjacent colours." This plausible statement suggests that the original map may be replaced if necessary by a simpler one, in which only the four specified countries are in contact with the central country. The rest of the proof quite correctly and ingeniously shows that this simpler map may be recoloured so as to use only 4 colours. Alas, it does not follow that the same recolouring necessarily succeeds with the original more complicated map. Suppose for example there were 6 countries in contact with the central country. We have focussed our attention on four of them originally all bearing different colours, and have shown that after recolouring only 3 colours are used on these 4 countries. But surely it is conceivable that the 4th colour is still used on one of the 2 countries so far ignored.

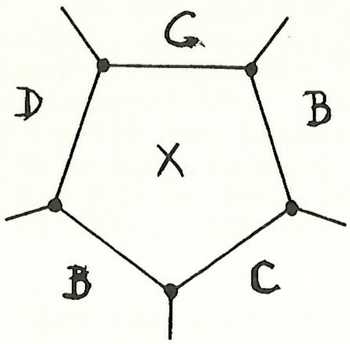
One can get apparently closer to a proof by using the well known fact that in every map there is at least one country which has no more than 5 neighbours. Using the technique of proof by mathematical induction we assume that every map with (n-1) countries can be coloured with 4 colours, and consider any map with n countries. Find a country X, with no more than 5 neighbours and colour the map consisting of the remaining (n-1) countries with 4 colours in accordance with the induction hypothesis. If X has 3 or fewer neighbours, it may obviously be coloured with one of the 4 colours already used. If X has 4 neighbours bearing different colours, Kuhn's argument shows how the colouring used may be changed so that now only 3 colours are used on X's neighbours.

It only remains to consider the situation when X has 5 neighbours, coloured as shown in Fig. 1. If the B-D chain starting at the country D_1 does not contain the country B_1, we can reverse the colours in it, thus replacing the colour D_1 with the colour B. Colour D can then be used for X.



However if the D-B chain starting at D_1 contains B_1, we would be no better off, so we leave the colours as they are, and consider instead the D-C chain starting at D_1. If this does not contain the country C_1 we reverse colours in it as before, replacing D on D_1 with colour C, and having D for use on X to finish colouring the map.

We are therefore finished unless the D-B chain starting at country D_1 includes country B_1 and also the D-C chain starting at D_1 includes country C_1. But in this case that B-D chain surrounds the A-C chain starting at country A_1 preventing it from reaching the country C_1. By reversing colours in this chain we can replace colour A on A_1 by C. Similarly the A-B chain starting at A_2 is surrounded by the C-D chain connecting C and D and on reversing its colours, we replace the colour A on A_2 by B. The colours around X are now as shown in Fig. 2, and X can be coloured with the colour A. Hence the n-country map can be coloured with 4 colours. This completes the proof of the induction step, and it easily follows that all maps can be coloured with 4 colours.



This was the "proof" of the 4-colour problem published by Kempe in 1879. It was 11 years before the English mathematician Heawood found a fallacy in it.

How long would you take? See page 40.

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<u>A scaly Cryptarithm</u>

Here is a cryptarithm with a difference. The following is the same addition problem using three different scales:

x	x	x
x	x x	x x
x	x x x	x x x
x	x x x	x x x x

Can you fill in the missing digits using only the digits 0, 1, 2, ... 9?

* * *

<u>3 for the box!</u>

Take 12 matches and lay them out end to end as a 3, 4, 5 right angled triangle. Now move (<u>not</u> remove!) 3 matches so that the area of the resulting figure or figures is 2/3 of the area of the original triangle.

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(Answer on page 40)