MORE FROM OUR READERS

In Parabola Vol 7 No 1, Question 3 in the Senior Division of the 1970 Mathematics Competition states:

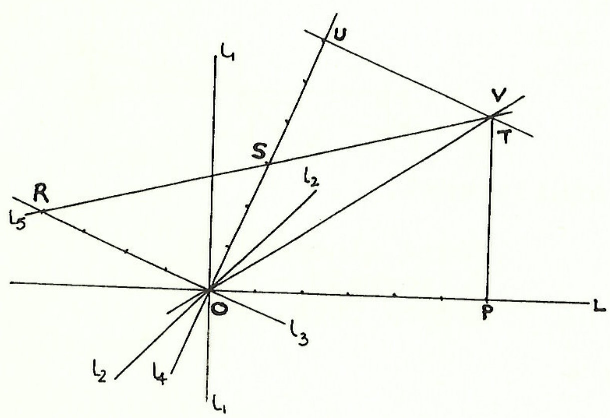
Suppose you are provided with a straight edge with two marks on it, one unit apart. The only operations allowable with this instrument are (a) ruling straight lines and (b) marking a point on a given straight line at a unit distance from a given point on the line. (You are not given a compass.) Using this instrument show how to perform the following constructions:

(i) Bisect a given angle.

(ii) Construct a square whose diagonals meet at a given point.

(iii) Construct a square with one vertex at a given point.

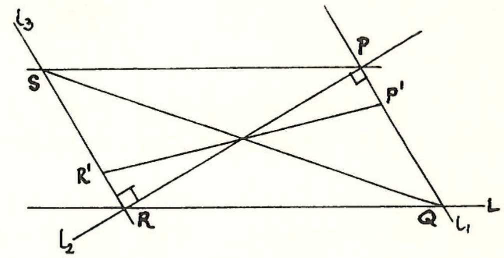
One way of doing this is as follows. Referring to the diagram, it is required to construct a perpendicular to the line L at point P. Mark 6 units off along L from P to get O. Through O draw any two lines, L_1 and L_2, and bisect the angles between them (using the method explained in the answer to question 3 part 1) to obtain L_3 and L_4 which are mutually perpendicular. Mark 4 units off along L_3 from O to get R, and three units from O along L_4 to get S. RS will then be 5 units long because LROS = 90°. Drawing L_5 through R,S and marking five units off along it from S away from R, obtain T, and obtain U by marking 3 units along L_4 from S. Find the bisector of LUOP and from V, the intersection of this line and UT draw a line to P. Now



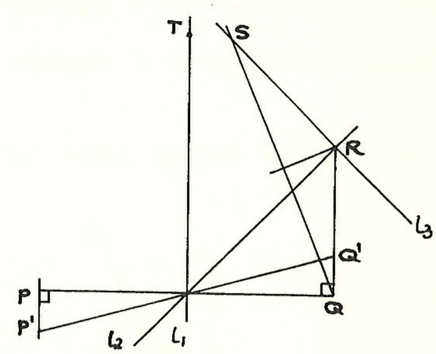
Therefore $\Delta ORS \equiv \Delta UTS$
and LSUV = 90°
therefore $\Delta OUV \equiv \Delta OPV$;
LOPV = 90°.

So PV is the required perpendicular.

This construction makes the construction of the perpendicular from a given point to a given line not passing through it relatively easy. Referring to the next diagram, let P be the point and L the line. Draw L_1 through P, intersecting L at Q, then L_2 through P, perpendicular to L_1, intersecting L at R. Then draw L_3 through R, perpendicular to L_2. Find P' one unit from P, R' one unit from R, lying on L_1 and L_3 respectively. Then draw a line through Q and the intersection of R'P' and L_2, to meet L_3 at S. SP is now parallel to RQ, so the perpendicular to SP at P is the required line.



The construction can also be used to make intervals with lengths of $\sqrt{2}$, $\sqrt{3}$ etc. units, and thus to construct figures such as the equilateral triangle, as in the next diagram, and regular pentagon, hexagon etc., given one side. Here PQ is the given side, which is bisected to give O as PR was above, L_1 is constructed perpendicular to PQ, this angle is bisected by L_2, which meets QQ' at R, the line perpendicular to L_2 at R is constructed (L_3), the angle between L_3 and RQ bisected, and the perpendicular from Q dropped onto this, and extended to meet L_3 at S. *OS is now $\sqrt{3} \times$ *OQ, and OS is transformed to OT just as RQ was transformed to RS. Thus ΔPQT is equilateral.



Yours sincerely,
N. Wormald
Sydney Tech. High

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Nicholas Wormald was the First Prize Winner in the 1971 School Mathematics Competition - Ed.

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Both Tony Holzherr and Barry Quinn sent in solutions to our "Trick" in the last issue. As Tony's arrived first we print his solution but follow it up with Barry's generalisation of the problem.

The article "A Trick" which appeared in Vol 7 No 3 page 21 of your magazine can be explained as follows:-

A trick Ask a friend to choose any number less than 1000. Then get your friend to tell you the remainders when he or she divides it by 7, by 11 and by 13. You will then be able to tell your friend the number he or she chose, after a few calculations, thus:

Multiply the first remainder by 715, the second by 364 and the third by 924. Add these together and reduce the answer to a least residue (see Vol 7 No 1) modulo 1001. This will be the number chosen.

Example Number chosen = 632. Remainders are 2, 5, 8. 2×715 = 1430; 5×364 = 1820; 8×924 = 7392. Total = 10642. Now divide by 1001 to get remainder 632.

Let the number chosen be x, therefore $x = 7a + r_1 = 11b + r_2 = 13c + r_3$ where a, b and c are integers and r_1, r_2 and r_3 are the first, second and third remainders respectively. By multiplying these remainders by 715, 364 and 924 respectively we obtain

$$715r_1 = 715x - 715\times 7a$$

$$364r_2 = 364x - 364\times 11b$$

$$924r_3 = 924x - 924\times 13c.$$

After adding,

$$715r_1 + 364r_2 + 924r_3$$

$$= 2003x - (5005a + 4004b + 12012c)$$

$$= x - (5005a + 12012c - 2002x).$$

We can now find $715r_1 + 264r_2 + 924r_3 \pmod{1001}$

$$= x - (5005a + 4004b + 12012c - 2002x) \pmod{1001}$$

$$= x \pmod{1001} - (5005a + 4004b + 12012c - 2002x) \pmod{1001}$$

(Note that $a - b \pmod{m} = a \pmod{m} - b \pmod{m}$.)

$$= x \pmod{1001}.$$

Since $x \leq 1000$, x must be the least residue. This explains the way the trick works.

Yours sincerely,
A. Holzherr
Cabramatta High.

Excerpt from Barry Quinn's letter

Let n = any number less than xyz (x, y and z given).
Let $n = xa + b = yc + d = ze + f$.
Now $b = n - xa$, $d = n - yc$, $f = n - 2e$.

The problem is to find numbers to multiply the remainders by. The simplest way is to have a complete xyz times something after expanding the expression. Therefore multiply b by yzg, d by xzh and f by xyi.

$$yzg(n-xa) + xzh(n-yc) + xyi(n-2e)$$
$$= n(yzg + xzh + xyi) - xyz(ag + hc + ie).$$

Now, $yzg + xzh + xyi$ has to be equivalent to 1 (mod xyz) so that n is left after reducing to a least residue (mod xyz).

Therefore
$$yzg + xzh + xyi = jxyz + 1$$
$$z(yg + xh) = xy(jz-i) + 1.$$

Let $X = yg + xh$, $Y = jz - i$.

$$zX = xyY + 1.$$

Now, x, y and z are known.

Therefore it is simple to find theleast values of X and Y: by trial and error. After finding these values, one can then find out the simplest values for g, h and i. You find out the values of g and h by the method shown on page 2 of Vol 5 No 1 Parabola, concerning Diophantine Equations.

An example of the generalisation which I have just proposed is:-

Let n be any number less than $5\times7\times9 = 315$.
yzg (the number to multiply the remainder after division by 5) = $63g$, $xzh = 45h$, $xyi = 35i$.

$$63g + 45h + 35i = 315j + 1$$
$$9(7g + 5h) = 35(9j-i) + 1.$$

Let $X = 7g + 5h$, $Y = 9j - i$.

$$9X = 35Y + 1.$$

The simplest values are $X = 4$, $Y = 1$, but g and h are whole numbers and so $7g + 5h \neq 4$. The next values are $X = 39$, $Y = 10$.

Therefore $7g + 5h = 39$.

$$5h = 39 - 7g$$
$$= 40 - 5g - (1+2g).$$

(Continued)

Therefore
$$1 + 2g = 5m$$
$$2g = 5m - 1$$
$$g = \tfrac{1}{2}(5m-1).$$
Therefore
$$5h = 39 - \tfrac{1}{2}(35m-7)$$
$$10h = 78 + 7 - 35m$$
$$10h = 85 - 35m$$
giving the only whole values for h and m
$$h = 5, \quad m = 1.$$

Therefore
$$7g + 25 = 39$$
$$g = 2.$$
Now
$$9j - i = 10.$$

The simplest value of i is 8, i.e. when $j = 2$. The numbers to multiply the remainders by are now respectively for the remainders of 5, 7 and 9,

$$63\times2 = 126, \quad 45\times5 = 225, \text{ and } 35\times8 = 280.$$

Yours faithfully,
Barry Quinn
St Joseph's College.

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$$\int \frac{d\ \text{cabin}}{\text{cabin}} = \log\ (\text{cabin}) + C$$
$$= \text{houseboat!}$$

Brian Martin
Cootamundra High.

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The following "Classroom Note" was published in the British Mathematical Gazette Vol LV No 391 of February 1971. It is reprinted here with their comment and that of Keith Burns. (He's older now!) - Ed.

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237. On note 194

The problem is: does x^2-1 have a factor of 24, when x is a prime number greater than 3? The factors of x^2-1 are $(x-1)$ and $(x+1)$. The prime number x is divisible by neither 2 nor 3. Hence either $(x+1)$ or $(x-1)$ has a factor 3. Both $(x+1)$ and $(x-1)$ are divisible by 2, and one of them has a second factor 2, making it divisible by 4. Thus x^2-1 has a factor 24.

Note: It is not necessary for x to be prime; it needs only to have no factor 2 or 3. Thus $35^2-1 = 24\times51$. (By Keith Burns (13), Campbell HS, Canberra ACT.)

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Also Keith Burns and Gabriel Holmik of De la Salle High School Kingsgrove, sent in solutions to "Find the Error" in Vol 7 No 2.

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And now, two problems from readers:-

In any 3 digit number divisible by 7, the cube of the number formed by the first two digits minus the cube of the third digit is divisible by 7. The same statement is true when either of 11 or 13 are substituted for 7.

Check this out - then prove the statement is always true.

Barry Quinn
St Joseph's College.

A suggestion from Brian Martin, Cootamundra High, arising from the title "Interesting Numbers" in Vol 7 No 2.

The numbers that are not interesting form a set S. Then surely the largest uninteresting number is interesting. Similarly the smallest element of S is interesting. Then the largest one left, say is interesting and so on. Finally we must reach the last remaining uninteresting number which must bethe most interesting of them all because if there is one, everyone will be trying to find something strange about it.

Conclusion: there are no uninteresting numbers.

(Get your criticisms in to me quickly before the School Competition so we can print a reply next issue. Incidentally, Brian Martin was second prizewinner in the Senior Division of the School Mathematics Competition 1971 - Ed.)

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<u>Correction to "Vertices Edges and Faces", Vol 7 No 3</u>

The torus shown in Diagram 4 on page 4 of Vol 7 No 3 unfortunately grew a handle while we weren't looking; the torus should <u>not</u> have a handle, the sphere should (and did).

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