SOLUTIONS

Solutions to Problems 161-170 in Vol 7 No 3

The names of successful problem solvers appear after the solution to Problem 170.

J161 Find a whole number, N, satisfying the following conditions:

(a) N is the product of exactly four distinct prime numbers.
(b) The decimal notation for its square is abc, def, abc where $a \neq 0$ and where the three digit number def is exactly twice the three digit number abc.

Answer $N = abc \times 10^6 + def \times 10^3 + abc$

$= abc(10^6 + 2.10^3 + 1)$ since $def = 2 \times abc$.

Now $10^3 + 1 = 1{,}001 = 7 \times 11 \times 13$, so, since N is the product of 4 distinct primes, abc is the square of a prime number. The primes, other than 11 and 13, such that $def = 2p^2$ is a three figure number, are 17 and 19. N must be either $17 \times 1{,}001$ or $19 \times 1{,}001$.

J162 Prove that for every positive whole number n,

$$(n+1)(n+2)(n+3) \ldots (2n-1)(2n) = 2^n.1.3.5. \ldots .(2n-1).$$

Answer LHS

$$= \frac{1.2.3.\ldots n(n+1)(n+2) \ldots 2n}{1.2.3 \ldots n}$$

$$= [1.3.5 \ldots (2n-1)] . \frac{(2.4.6 \ldots 2k \ldots 2n)}{1.2.3 \ldots k \ldots n}$$

$$= [1.3.5 \ldots (2n-1)] . \underbrace{2.2 \ldots 2}_{n \text{ factors}}$$

$= RHS.$

J163 Given the equations $x/y = x-z$, $x/z = x-y$ prove that $y = z$ and that x is not between 0 and 4.

Answer Note that $x = 0$ is impossible since we readily obtain y or $z = 0$ and division by zero is necessarily involved. Clearing of fractions gives $x = xy - yz$ and $x = xy - yz$ whence $xy = xz$ and

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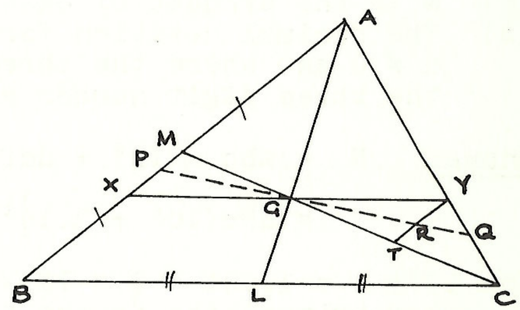
$y = z$. Putting $y = z$ in the first equation yields $x = xy - y^2$, whence $x = y^2/(y-1) = 4 + (y-2)^2/(y-1)$.

The first expression for x shows that $x < 0$ when $y < 1$, $(y \neq 0)$ and the second shows that $x \geq 4$ when $y \geq 1$. Since there is no solution if $y = z = 1$, x never lies between 0 and 4.

0164 A triangle is divided into two figures by a straight line through the centroid. Show that the ratio of the area of the smaller piece to that of the larger may take any value between a minimum of 4/5 and a maximum of 1.



Answer In the figure, XY is the line through the centroid G which is parallel to the base BC. The ΔAXY is similar to ABC, being reduced by the linear factor 2/3 (remember that G trisects the median AP). Hence area of ΔAXY = 4/9 Area of ΔABC so that area of BCYX = 5/9 area of ΔABC and the ratio of the areas is 4:5.

CM is a median of ΔABC. Area of ΔAMC = area of ΔBMC (equal bases AM and BM, same height). Hence ratio of areas is 1:1.

If P is any point on the interval XM, draw the straight line PG cutting AC at Q. We are finished if we can show that 4/9 ΔABC < area of ΔAPQ < ½ ΔABC.

Construct YRT parallel to AB cutting PQ at R and MC at T.
Then ΔPGM ≡ ΔRGT and ΔPGX ≡ ΔRGY.
Hence area ΔPGM < area ΔQGC
and area ΔPGX < area ΔQGY.

Therefore ΔAPQ = ΔAXY + ΔQGY − ΔPGX > ΔQXY = 4/9 ΔQBC
and ΔAPQ = ΔAMC + ΔPGM − ΔQGC < AMC = ½ ΔQBC.

(NOTE: ≡ means "is congruent to", = means "equals in area".)

The same argument applies for other lines through G, but using a different median and/or a line through G parallel to a different side of the triangle ABC.

0165 There are two prime numbers p such that $(2^{p-1}-1)/p$ is a perfect square. Find them, and prove they are the only ones.

Answer The values of p are 3 ($2^2-1 = 3.1^2$) and 7 ($2^6-1 = 7.3^2$). By inspection $p \neq 2$.

Now if $2^{p-1}-1 = py^2$

$$(2^{\frac{1}{2}(p-1)}-1)(2^{\frac{1}{2}(p-1)}+1) = py^2.$$

The factors $(2^{\frac{1}{2}(p-1)}-1)$ and $2^{\frac{1}{2}(p-1)}+1)$ differ by 2 and, as they are both odd, are relatively prime. We must have one of them equal to y^2, the other equal to $py_2{}^2$ where $y_1y_2 = y$.

Now $2^{\frac{1}{2}(p-1)}-1$ is the square of an odd number only if p = 3, since, for larger values of p, $(2^{\frac{1}{2}(p-1)}-1)$ is one less than a multiple of 4, whereas every odd square is one more than a multiple of 4. In fact

$$(2k+1)^2 = 4k(k+1) + 1.$$

Similarly, $(2^{\frac{1}{2}(p-1)}+1)$ is an odd square only for p = 7. Solving $4k(k+1) + 1 = 2^{\frac{1}{2}(p-1)}+1$ gives

$$k(k+1) = 2^{\frac{1}{2}(p-5)}.$$

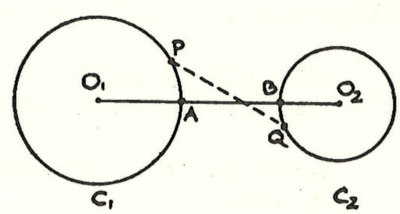
The LHS is the product of two consecutive integers and must contain an odd prime factor except when k = 1.

0166 C_1 and C_2 are any two non-intersecting circles. P is any point on C_1, Q any point on C_2. Prove that the distance PQ is least when both P and Q lie on the line of centres. You should consider the two cases: C_1 internal to C_2, C_1 external to C_2. The shortest and most elegant proof is sought.

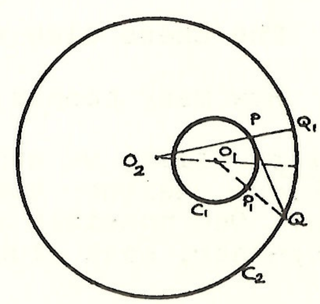
Answer (i) C_1 external to C_2. Join the centres O_1 and O_2 by a straight line cutting the circles at A and B. Let P and Q be any two points on C_1 and C_2 respectively. Since a straight line is the shortest distance between points

$$O_1A + AB + BO_2 < O_1P + PQ + QO_2.$$

Observing that $O_1A = O_1P$ and $BO_2 = QO_2$ it follows that $AB < PQ$.



Answer (ii) Let PQ be the shortest distance with P on the smaller circle C_1 and Q on the outer circle. If O_2 is not on PQ (produced) let O_2P produced cut C_2 at Q_1. Then $O_2P + PQ > O_2Q = O_2Q_1 = O_2P + PQ_1$ whence $PQ > PQ_1$ contradicting the



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the "shortest distance" definition of PQ. Hence O_2 lies on QP.

Similarly if O_1 does not lie on PQ, let O_1Q meet C_1 at P_1. We have $O_1P + PQ > O_1Q = O_1P_1 + P_1Q$ whence $PQ > P_1Q$ another contradiction. Hence both O_1 and O_2 lie on QP.

0167 Show that for the angles A, B and C of a triangle,

$$\tan^2 \tfrac{1}{2}A + \tan^2 \tfrac{1}{2}B + \tan^2 \tfrac{1}{2}C \geq 1.$$

Answer For any 3 angles x, y, z,

$$\tan (x+y+z) = \tan [(x+y) + z]$$

$$= \frac{\tan (x+y) + \tan z}{1 - \tan (x+y) \tan z}$$

$$= [\frac{\tan x + \tan y}{1 - \tan x \tan y} + \tan z] \div [1 - \frac{(\tan x + \tan y)\tan z}{1 - \tan x \tan y}]$$

$$= (\sum \tan x - \tan x \tan y \tan z) \div (1 - \sum \tan x \tan y).$$

As $\cot (\sum \tfrac{1}{2}A) = 0$, we get $\sum \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = 1$.

Using $\sum (\tan \tfrac{1}{2}A - \tan \tfrac{1}{2}B)^2 \geq 0$ we obtain $\sum \tan^2 A \geq \sum \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B$

$= 1.$

Clearly equality holds only when $\tan \tfrac{1}{2}A = \tan \tfrac{1}{2}B = \tan \tfrac{1}{2}C$; i.e. when ABC is equilateral.

0168 The following observations were made at a meeting of delegates of clubs at a school:

1. For every pair of clubs represented there was exactly one person present who was a member of both.

2. Each person present was a member of at least two different clubs.

3. The chess club sent exactly three delegates.

How many people were present? How many clubs were represented?

Answer Alas, we appear to have "goofed" rather badly with this problem. Neither question can be uniquely answered from the given data. One possible situation would consist of 4 clubs and 6 people altogether, each club being represented by 3 delegates.

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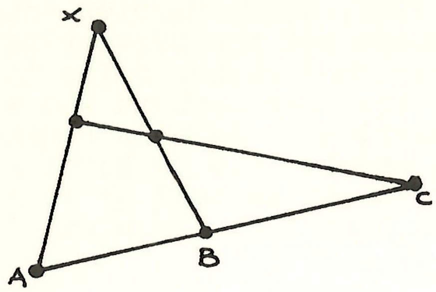

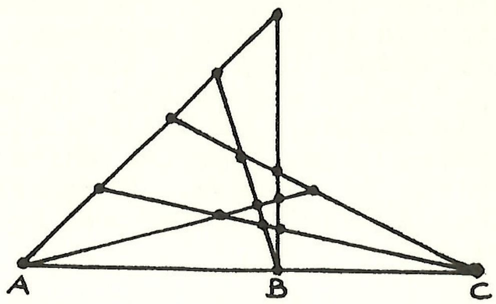

Figure 1 is intended to illustrate this situation, each intersection representing a person, and the three intersections along a line the delegates from one club.

Figure 2 illustrates another possible solution with 7 clubs and 15 people. {A,B,C} is the chess club. All the other clubs are represented by five delegates. There are, in fact, infinitely many possible arrangements satisfying the given conditions.

<u>0169</u> A rectangular sheet of paper is ruled off into m horizontal rows and n vertical columns. On each of the mn squares so formed is marked either a nought or a cross in such a manner that each row and column contains both noughts and crosses. Prove that there are four squares, a, b, c, d, such that a, b lie in one column and c, d in another while a, c lie in one row and b, d in another row (i.e. they are the corners of a possibly smaller rectangle) and, either a and d are crosses and b and c are noughts, or a and d are noughts and b and c are crosses.

<u>Answer</u> Let k ($< m$) be the maximum number of crosses occurring in a column and let c_1 be a column with k crosses in rows $a_1, a_2 \ldots a_k$. Let c_1 have a nought in row r_1 and let r_1 have a cross in column c_2. If c_2 had crosses in each of the rows $a_1, a_2, \ldots a_k$ as well as r it would contain k+1 crosses, contradicting the definition of k. Hence c_2 has a zero in at least one of the rows $a_1 \ldots a_k$, and we select this row for r_2.

The four squares at the intersection of columns c_1, c_2 with rows r_1, r_2 may be appropriately labelled a, b, c, d to satisfy the requirements listed in the question.

0170 This puzzle in pure logic is due to Raymond Smullyan of Princeton University. On Christmas Day 1970, three married couples celebrated by dining together. Each husband was the brother of one of the wives; that is, there were three brother-sister pairs in the group. Helen was exactly 26 weeks older than her husband who was born in August. Mr White's sister was married to Helen's brother's brother-in-law. She (Mr White's sister) married him on her birthday which was in January. Marguerite White was not as tall as William Black. Arthur's sister was prettier than Beatrice. John was fifty years old.

What was Mrs Brown's first name?

Answer Using statements 1 and 5 only we see that the people present are represented by one or other of the two diagrams A and B.

	Mrs Brown	Mrs Black	Marguerite White
A			
	Mr Brown	William Black	Mr White
	Mrs Brown	Mrs Black	Marguerite White
B			
	Mr Brown	William Black	Mr White

where the lines indicate brother-sister relationships.

Let us consider A first. There are two ways of assigning the first names. Arthur and John to the surnames Brown and White.

Case A_1 Arthur White and John Brown. Using 6 we see that Mrs Brown is not Beatrice so the women are Helen Brown and Beatrice Black. This situation does not conflict with statement 3, but it does conflict with 2, 4 and 7. Even if Helen's birthday was on January 31st 1920 and John's on August 1st, she is 26 weeks + 1 day older than he is (remembering that 1920 was a Leap Year).

Case A_2 Arthur Brown and John White. Using 6, the women are Beatrice Brown and Helen Black. This is in conflict with 3 since Arthur Brown is not his own brother-in-law.

Case B_1 Arthur White and John Brown. Using 6, the women are Beatrice Brown and Helen Black. No conflict is found with any statement.

Case B_2 Arthur Brown and John White. If the women are Helen Brown and Beatrice Black, statement 3 is not true. Therefore the women's

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names must be Beatrice Brown and Helen Black.

Hence in both of the allowable solutions Mrs Brown's first name is Beatrice.

* * *

<u>Solvers of Problems 161-170</u>

Robert Kuhn, Sydney Grammar - 161, 162, 163, 164, 168-70, and proved half of 166.
Peter Corkish of Belmore Boys' High solved 166.
Barry Quinn of Marist Bros Hamilton solved 166 and also got the correct answer to 170. His reasoning was incomplete, however, for 170.

Names of late solvers will be published in Vol 8 No 2.

<u>Late solvers of Problems 151-160</u>

Keith Burns, Campbell High ACT - 154-60. Keith's solution to Problem 155 was outstanding and of course led to his article in this issue.
Kevin Thompson of Kingsgrove North High - 156.

In addition, the mysterious "H de F" mentioned in Vol 7 No 3 as a solver turned out to be Henri de Ferandy of St Joseph's. Félicitations, Henri.

C.D. Cox
Problem Editor.

* * *

<u>Correction to "Pythagoras and all that"</u>

My remarks about Fermat's Last Theorem should have pointed out that there are two cases to consider:

(i) n has no factors in common with a, b or c,

(ii) n has some factor in common with at least one of a, b or c.

In the article I discussed (i). Case (ii) is much more difficult to handle and the number of n for which the theorem has been proved true under condition (ii) is far less than in case (i).

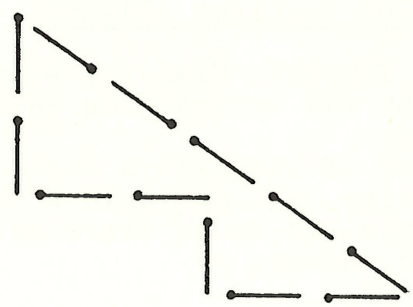
M.G. Greening.

ANSWERS

The 4 Colour Map Problem

The fallacy in Kempe's "proof" of the 4 colour problem:
The trouble is that the B-D chain and the C-D chain starting at country D can cross at a country coloured with colour D, and need not completely isolate the A-C chain starting at A_1 from the A-B chain starting at A_2. Colour reversals in both these chains could then leave two adjacent countries (previously coloured B and C) now both coloured A; which of course is not allowable.

3 for the box!



Area removed = 2 "square matches".

Do it in your head!

Their relative speeds (i.e. the speed one train seems to the other train to be travelling) is 180 km per hour or 3 km per minute. Thus they are 3 km apart one minute before they cross.

"Eight"

10,020,316 ÷ 124 = 80,809.