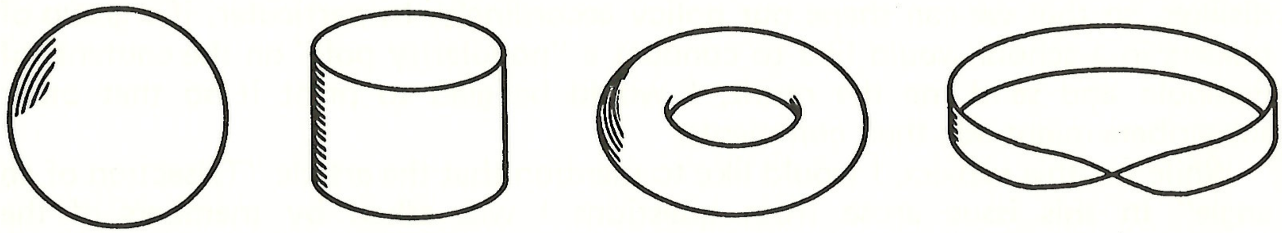
CUTTING THE MOBIUS STRIP

In Figure 1, there are four different surfaces: the sphere, the cylinder, the torus and the Mobius strip. I expect you are familiar with the first two of these. The torus is the name of the surface which is like the inner-tube of a rubber tyre.



Sphere Cylinder Torus Mobius Strip

Figure 1.

I mean the surface to be hollow like an inner tube, not solid like a doughnut. In the same way, I mean the sphere to be a hollow sphere like a ping pong ball, not solid like a billiard ball. The last surface, the Mobius strip, may be completely new to you. It looks like a cylindrical strip with a twist in it. You can make a Mobius strip from an ordinary strip of paper in much the same way as you would make a cylinder. Figure 2 shows the method; you take the two ends of the strip marked "a" and twist them so that the arrows marked on the ends point in the same direction, and then stick the ends together.

The Mobius strip exhibits some peculiar properties that the cylinder does not. For example, Figure 3 shows that if you cut along a line drawn around the middle of the Mobius strip it stays in one piece. This is probably the most striking property of the strip. Why does it happen? One way of finding out would be to study the Mobius strip itself very carefully. However, the mathematician usually employs another method to discover the answer to such a question. He looks for the same kind of behaviour somewhere else in the hope that when he sees it there, it may be easier to explain. In this case, one should look at other surfaces to see if it is ever possible to cut along a closed path without having the surface fall into two pieces.

If you examine the sphere you will very soon come to the conclusion:—

> "It is impossible to cut around a closed circuit on a spherical surface without dividing the surface into two separate parts."

On the other hand, if you examine the torus, as in Figure 4, you can discover two types of closed paths along which you can cut without having the torus come

Cylinder | Mobius strip

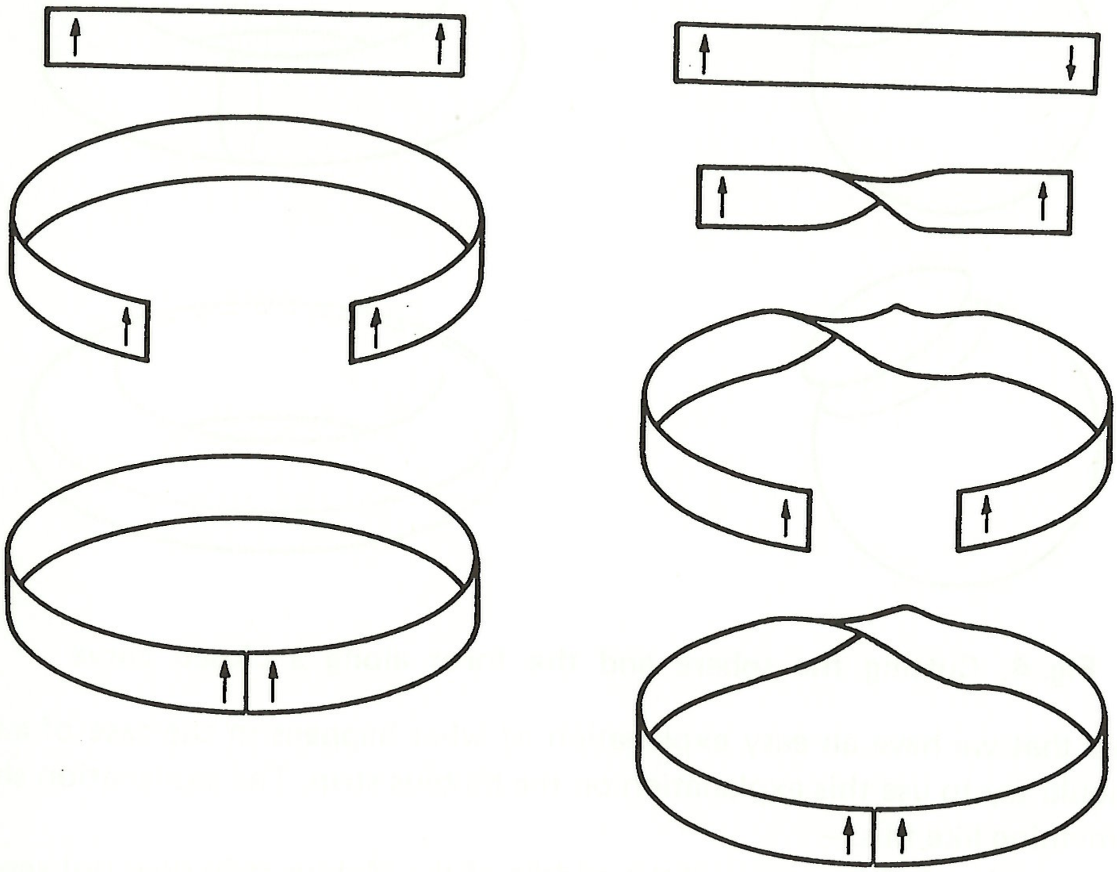
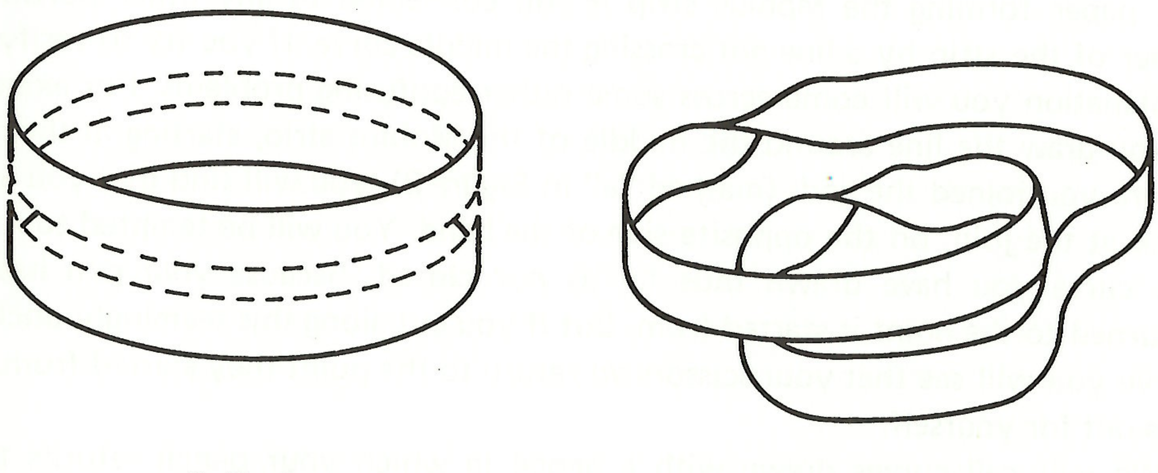


Fig. 2. Steps for making the cylinder and Mobius Strip



Cylinder | Mobius strip

apart. This is the phenomenon we should be looking for. Why doesn't the torus fall into two pieces when you cut around the curve, as in the top drawing of Figure 4? This is easy to explain. The curve doesn't separate any part of the torus from any other part. When the torus is cut along this curve, one can still get from any one part of the torus to any other without having to cross the gap.

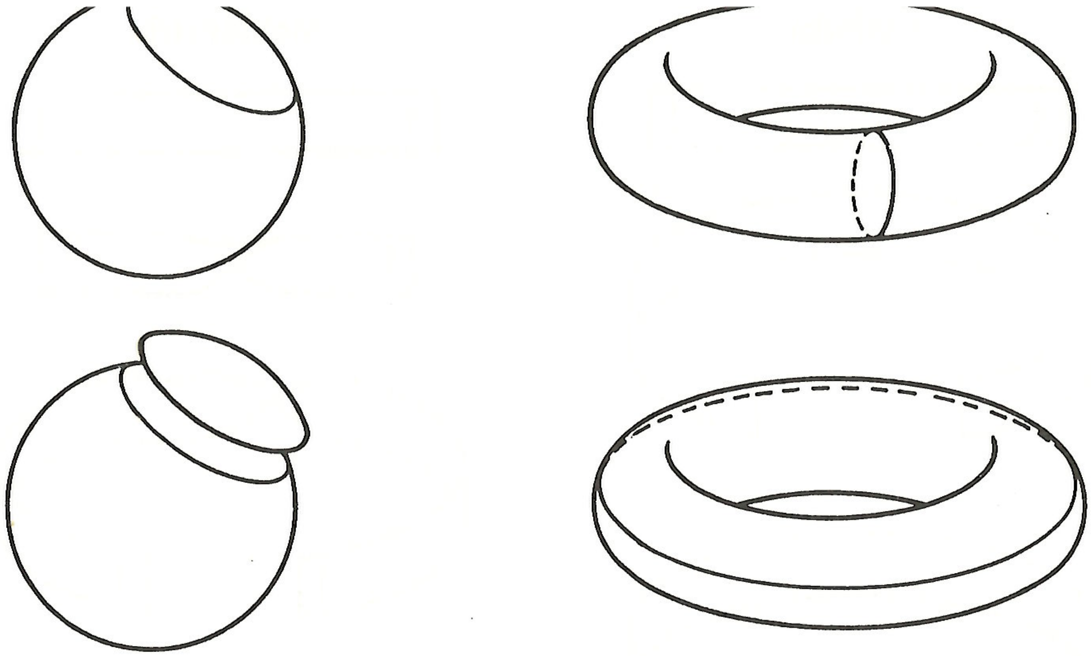


Fig. 4. Cutting the sphere and the torus along a closed curve

Now that we have an easy explanation of what happens in the case of a torus we should try to use this explanation on the Mobius strip. The explanation should go something like this:—

The closed path drawn around the middle of the Mobius strip does not separate any one portion of the strip from any other. I mean by this that any portion of the paper forming the Mobius strip is still connected to any other portion of paper of the strip by a line not crossing the middle curve. If you try to verify this explanation you will come across some rather confusing problems. For example, if you draw the line around the middle of the Mobius strip, starting at the place where you joined the ends (marked "a" in Figure 2), you will find that you arrive back at the join, on the opposite side of the band. You will be tempted to regard the curve you have drawn thus far as *not* closed, because your pen has not returned to the point it started from. But if you cut along this seemingly unclosed curve you will see that your scissors *do* return to the point they started from. Try this out for yourself.

We will call curves drawn with a pencil in which your pencil returns to its actual starting point (on the same side of the paper), "pencil-closed". The kind just referred to where we return to the opposite side of the paper, we will call "scissor-closed". To draw curves which are "scissor-closed" I should use a dye that goes through the paper so that I would be drawing on both sides of the paper at once. Notice that if you did this "dye drawing" on the cylinder you would simply be drawing two curves, one on the inside and one on the outside of the cylinder, simultaneously. The same is true of the sphere and the torus. This shows that on the sphere, torus and cylinder, "pencil-closed" curves are the same as "scissor-closed" curves, but on the Mobius strip they are not.

To show that the middle curve (drawn with the dye) doesn't separate any one part of the Mobius strip from any other, you have only to pick two points on the Mobius strip and connect them with a dye-drawn curve. Since the dye penetrates the paper, the dye-drawn curve reflects accurately which portions of actual paper are joined to what other portions. The clearest test of this idea is to mark two points on opposite sides of the middle curve with dye which goes through to the other side of the strip. Now, starting at one point, travel around the strip parallel to the middle curve. As you travel along you will find that because of the twist in the Mobius strip, you will arrive at the place where dye from the second point has come through the paper.

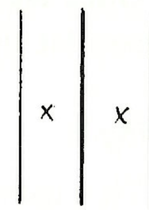


Fig. 5

Perhaps another way of experimenting with drawing curves on a Mobius strip is to make a strip out of thin, transparent plastic. In this case, no matter which side of the strip you actually drew on, the curve could be seen from both sides at once. It would be very easy then to demonstrate the explanation as to why, when one cuts along the middle curve on the Mobius band, the band doesn't separate into two pieces.

You might like to experiment with other surfaces in the following ways:—

1. Can a closed curve be drawn on the surface which doesn't separate it into two pieces?
2. How many different kinds of closed curves are there on the surface which don't separate it into two pieces?

For the first experiment, one could look at the conclusion we drew, rightly or wrongly, about closed curves on a sphere. (a) Is it correct? (b) Are there other surfaces for which this hypothesis appears to be true? (c) Can one show that these surfaces are related in any way? (i.e. if the hypothesis is true for one of them, must it be true for all of them?)

As for the second experiment the torus and Mobius band provide lots of examples. The study of these problems is called Topology.

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