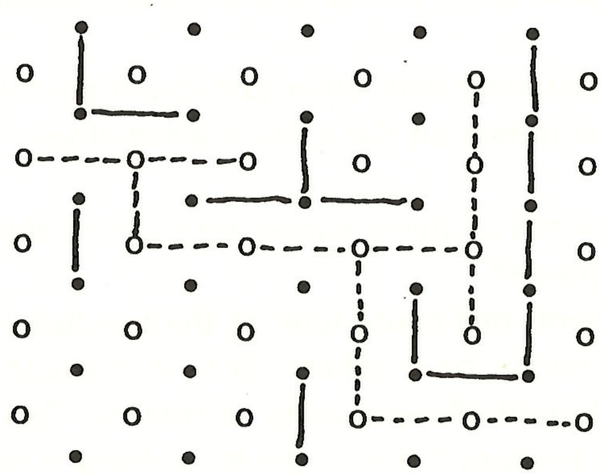
MATHEMATICAL GAMES

Once again we are presenting a game for you to play (and maybe to analyze). Please let us know what you think of the games and whether you want any more.

Bridg-it

This game was invented by David Gale of Brown University in America about fifteen years ago, and is now sold as a board game in America. To play Bridg-it, draw a number of black dots in a rectangular array interspersed with the same number of red dots, as in the diagram below (where 30 black dots and 30 red dots have been used and where open dots represent the red dots and solid dots represent the black dots). One player takes a black pen and joins two black dots vertically or horizontally. The other player, using a red pen then joins two red dots either vertically or horizontally. (The game is something like "boxes"). The game continues, each player joining two of his dots, until one player joins the opposite sides of the board with his colour. No lines are allowed to cross, so you can block your opponent.

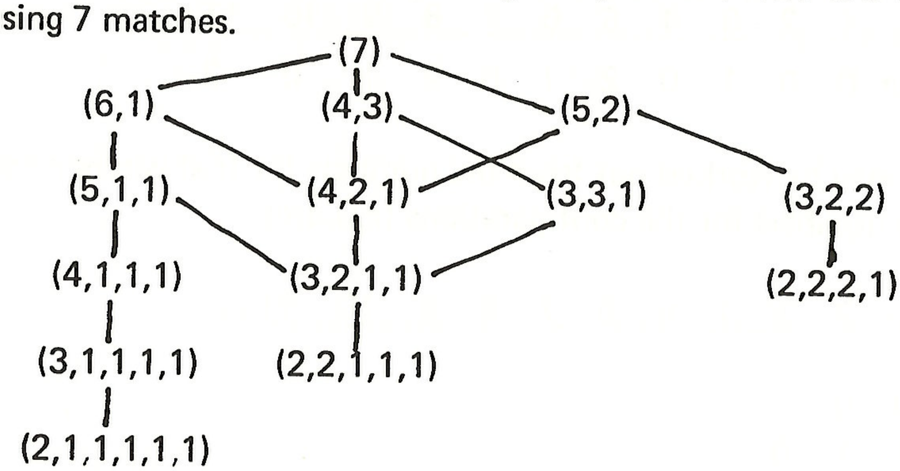


A game of Bridg-it. Red has won. (Dotted lines represent red lines.)

It has been proved that one of the players can force a win. Which one? Some of the keener readers might like to try to work out a winning strategy.

Grundy's and Kenyon's Games

The strategy for winning these two games which were described in the last issue (Vol 8 No 1), depends on the idea of "safe" and "unsafe" configuration. We will call a collection of heaps (such as (4,2,2,1) or (3,2,2,1,1) in the example last time) a *configuration*. One of these configurations is said to be *safe* if the next person to move can force a win, and is called *unsafe* if the next person to play either cannot move or is forced to move to a safe configuration (and so the other player can then force a win). For example, any configuration with heaps of only 1 or 2 is unsafe since the next person cannot move, but the configurations (3,2,2,1,1,1) and (5,1) are safe for Grundy's game since (5,1) → (4,1,1) → (3,1,1,1) → (2,1,1,1,1) are the moves for the next person to win. One way of deciding the correct tactics is to draw a tree of the possibilities (as you do in Probability). We will do this for Grundy's game and let you do it for Kenyon's game yourself. Here is a tree for Grundy's game using 7 matches.



So, in this case, the second player can force a win by moving to (4,2,1).

A quick way to decide whether a given configuration is safe or unsafe is to give each of the configurations on the tree a number. A dead-end is always given the number 0. Any other configuration is given the smallest number which is different from the number or numbers given to the configurations only one move away. Thus, on our tree, we will give (3,1,1,1,1), (3,2,1,1) and (3,2,2) the number 1, while (4,1,1,1), (4,2,1) and (3,3,1) are all given the number 0. (This last numbering is allowed because all the configurations which follow (4,1,1,1), (4,2,1) and (3,3,1) have the number 1.) (5,1,1) is given the number 2 (0 is used up by (4,1,1,1) and 1 is used up by (3,2,1,1)); similarly, the configurations (6,1) and (4,3) have the number 1, (5,2) has the number 2 and (7) has the number 0. A configuration with the number 0 is unsafe and any other configuration is safe. Consequently, the strategy is always to move to a configuration which has the number 0; i.e. the second player can win by moving to (4,2,1) as we said before.

There is a nice little trick we may use to work out the numbers of any configuration with more than one heap: write down the numbers for each heap using the base 2 and add them up, forgetting to "carry" any digits. The answer is then the number (with base 2) of the configuration.

e.g. (1) the number for (5) is 2 = 1 0
the number for (3) is 1 = 1
the number for (5,3) is 1 1 = 3

(2) the number for (8) is 2 = 1 0
the number for (5) is 2 = 1 0
the number for (8,5) is 0 0 = 0

The numbers for single heaps (n) are:

n	1	2	3	4	5	6	7	8	9	10	11 . . .
number	0	0	1	0	2	1	0	2	1	0	2 . . .

These may be worked out from the numbers for (r,s) where r + s = n.

The numbers for the configurations (r,s) are:

r\s	1	2	3	4	5	6	7	8	9	10	11 . . .
1	0										
2	0	0									
3	1	1	0								
4	0	0	1	0							
5	2	2	3	2	0						
6	1	1	0	1	3	0					
7	0	0	1	0	2	1	0				
8	2	2	3	2	0	3	2	0			
9	1	1	0	1	3	0	1	3	0		
10	0	0	1	0	2	1	0	2	1	0	
11	2	2	3	2	0	3	2	0	3	2	0

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A similar table can be made up for Kenyon's game.

R. James

Extra Note on Kenyon's Game

In the case of Kenyon's game, we can get a much easier strategy using a *pair* of numbers, instead of a single number, for each configuration, as follows:—

Heaps of 1, 2 or 3	(0,0)
Heaps of 4	(1,0)
Heaps of 5, 9, 13, 17 . . .	(0,0)
Heaps of 6, 10, 14, 18 . . .	(0,1)
Heaps of 7, 11, 15, 19 . . .	(1,0)
Heaps of 8, 12, 16, 20. . .	(1,1)

A pair of numbers is now assigned to each configuration by taking its first number to be the sum of the first numbers of its constituent heaps and its second number to be the sum of the second numbers of its constituent heaps. For example, the configuration (15,28,49) has the pair (1,0) + (1,1) + (0,0) = (2,1). More examples are given next to the specimen game below.

Suppose a game starts with heaps of 5, 8 and 9, and player A has first move. Then:—

A moves to 5, 3,5, 9.	(0,0)
B moves to 1,4, 3,5, 9.	(1,0)
A moves to 1,4, 3,5, 2,7.	(2,0)
B moves to 1,1,3,3,5, 2,7.	(1,0)
A moves to 1,1,3,3,5, 2,2,5.	(0,0)
B moves to 1,1,3,3,1,4, 2,2,5.	(1,0)
A moves to 1,1,3,3,1,4, 2,2,1,4.	(2,0)
B moves to 1,1,3,3,1,1,3,2,2,1,4.	(1,0)
A moves to 1,1,3,3,1,1,3,2,2,1,1,3.	(0,0)

As there are no more moves left, A is the winner.

The reason for his win is to be found in the fact that all configurations are safe *except* those which have each number of their pairs even. Thus, above (5,3,5,9) or (1,4,3,5,2,7) are unsafe but (5,8,9), the initial configuration, or (1,4,3,5,9) are safe. Examination of the rules for assigning pairs of numbers should convince you of the truth of the following two statements:—

(a) Any move that is possible from an unsafe configuration gives a safe one.

(b) There is always a possible move from a safe configuration to an unsafe one.

Bear in mind that when no move is possible from an unsafe configuration, the player concerned has lost, and also remember that heaps of 1, 2 or 3 cannot be split further as there must be a difference of at least 2 between the two new heaps.

Your strategy should now be clear. If you face a safe configuration you move so as to leave your opponent with an unsafe one. On the other hand, faced with an unsafe configuration yourself, you can only hope that somewhere in the game your opponent will be foolish enough to move from a safe configuration to another safe one. Unlike noughts and crosses, a knowledge of the correct strategy by both players will not, and cannot, lead to a draw!

P. Donovan

Readers might like to work out the connection between the pairs of numbers in the above note and the "Grundy" numbers for Kenyon's game — it's obvious when you see it!

Solution to Cross Number in Vol 8 No 1

7	9	3	5	1
1	■	1	1	2
9	7	■	2	7
2	2	5	■	8
■	9	6	6	9

We regret that (i) this was printed incorrectly with the bottom right hand corner filled in instead of being left blank. (ii) Our correction sheet, sent out to individual subscribers, while allowing a solution, was itself not completely accurate. Both 4 down and 11 across were originally intended to "share" the bottom right hand corner as in the solution above: i.e. 4 down is "12789", 11 across is "9669".

Solvers The first correct solution received was sent in by A. Fekete, 1st Form, Sydney Grammar School. Other correct solutions were sent in by Jeff Holten, Class 3A, East Hills Boys High School, and by David Paterson (age 14) of South Sydney Boys High School.