TRISECTION OF AN ANGLE

This question seems to be creating some interest among our readers. It cannot be done by traditional geometric means, that is, by compasses and straight edge, because while these methods enable us to draw parallels so as to divide line segments in given rational ratios, or to construct square roots and square roots of sums and differences of square roots etc, they can never enable us to find cube roots or solve cubic equations, in general. We can certainly divide 180° into 60° angles, and 90° into 30° angles, but this last we really do by bisection, or by using the specific property of the 60°–30°–90° triangle. We cannot trisect 60° into 20° angles.

If we put this into terms of coordinate geometry, compasses and straight edge enable us to construct lines: $y = mx + c$, and circles: $x^2 + y^2 + 2gx + 2hy + c = 0$, where the m, c, g, h are rationals or particular kinds of surds which arise from the intersections of lines and lines, lines and circles, or circles and circles. Remember, we start with a unit length that we reduplicate to give the positive integer lengths, and that we can then divide these to give us the positive rational lengths. The surds arise from the construction of circles.

You may be feeling that this is all rather irrelevant, as we are interested in the trisection of angles, not in finding cube roots of numbers which represent lengths. However, we will now show you that if you could trisect an angle — any angle — by ruler and compasses, this would also provide a solution to a cubic equation whose roots represent lengths of line segments or ratios of lengths.

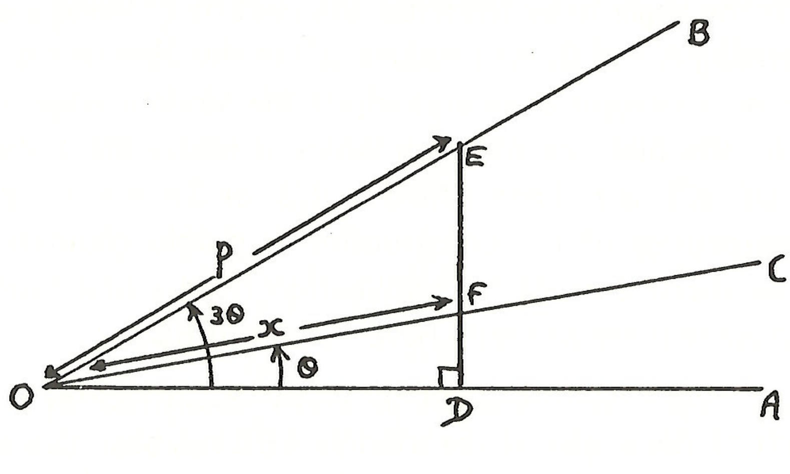

Diagram 1

If $\angle AOB = 3\theta$ and $\angle AOC = \theta$, we may step off the distance 1 unit along OA to get to D. At D we erect a perpendicular to OA to cut OC at F and OB at E. Let OF = x and OE = p. (See Diagram 1). Then $\cos\theta = 1/x$, $\cos 3\theta = 1/p$. Substitute in the trigonometric identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ to get $p^{-1} = 4x^{-3} - 3x^{-1}$ or $x^3 + 3px^2 - 4p = 0$. It can be proved that the roots of this last equation, in general, are neither rational, nor surds, nor sums of surds.

However, if we allow methods similar to that used in the construction of $\sqrt[3]{2}$ in our last issue, we can trisect angles.

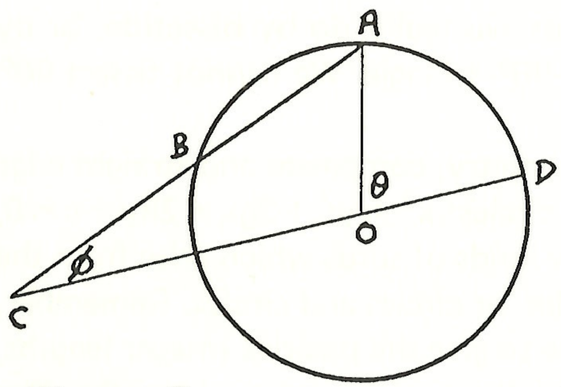

Diagram 2

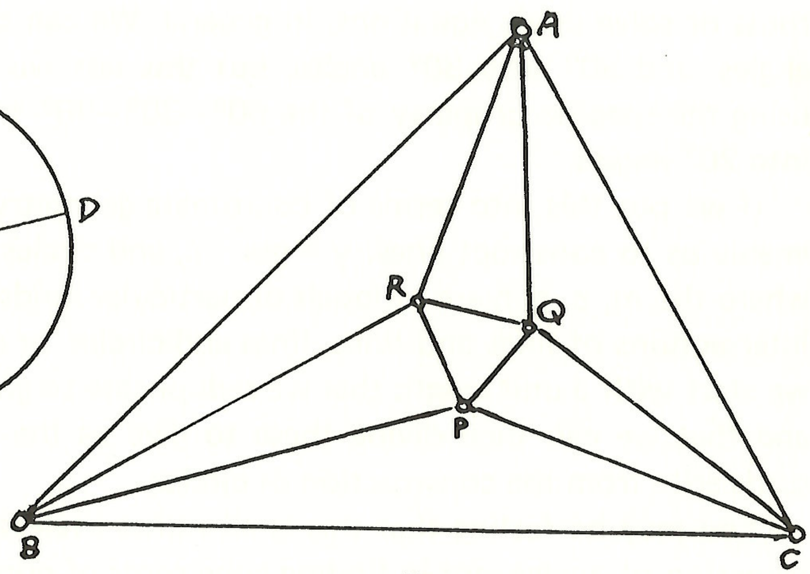

Diagram 3

In Diagram 2, θ is the angle to be trisected. We start by drawing a circle centre O, radius = r, thus getting A and D. We produce DO in the direction of C. We now mark the distance r on a straight edge and adjust the straight edge so that (i) it passes through A, (ii) the distance between where it meets the circle again (B), and where it meets DO (C), is r. Then $\angle ACD = \theta/3$, or $3\varphi = \theta$ on the diagram. The proof of the correctness of the construction is purely geometric and very straightforward. (How about a proof for publication from a first-former? -- Ed.) It is also as easy to show that we will only have $\theta = 3\varphi$ if BC = r.

In conclusion, you might be interested to know about Morley's triangle. In Diagram 3 above each of the angles of the triangle ABC has been trisected. ΔPRQ is always an equilateral triangle (Morley's triangle), no matter what the angles A, B, C are. Can you prove it?