SOLUTIONS TO SCHOOL MATHEMATICS COMPETITIONS 1972

JUNIOR DIVISION

Question 1

(i) Find two positive integers x,y so that $xy = x + y$.

(ii) Show that the positive integers you have found are the only ones which solve the equation.

(iii) If we have two positive integers x,y such that $x + y > xy$, what can be stated about x or y or both?

Answer

(i), (ii). $x(y-1) = y$, so x divides y. Similarly we can show y divides x. This gives $x = y$ and thus $y - 1 = 1$, $y = 2$, $x = 2$. Alternatively, we may divide by xy to get $1 = \frac{1}{y} + \frac{1}{x}$. We cannot have $x = 1$ or else $\frac{1}{y} = 0$, so $\frac{1}{x} \leqslant \frac{1}{2}, \frac{1}{y} \leqslant \frac{1}{2}$. This, with $\frac{1}{x} + \frac{1}{y} = 1$ demands $x = 2$, $y = 2$.

(iii). This gives $\frac{1}{x} + \frac{1}{y} > 1$. By the above, one, at least, of x and y must be 1.

Question 2

A right angled triangle has sides of length a, b and c, where c is the length of the hypotenuse. Prove that for any integer $n > 2$,

$$c^n > a^n + b^n.$$

Answer

Suppose $c^k \geqslant a^k + b^k$ for $k \geqslant 2$.

$$\begin{aligned}\text{Then } c^{k+1} &= c.c^k \\ &\geqslant c(a^k + b^k) \\ &\geqslant ca^k + cb^k \\ &> a^{k+1} + b^{k+1} \text{ since } c > a, c > b.\end{aligned}$$

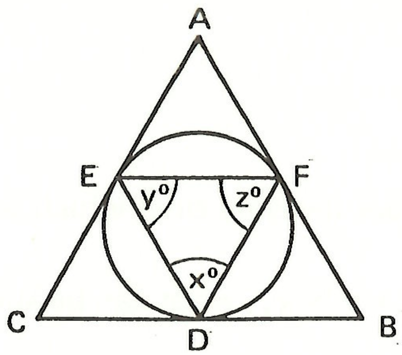
Since $c^2 = a^2 + b^2$, it follows that $c^3 > a^3 + b^3$ and thus $c^4 > a^4 + b^4$, $c^5 > a^5 + b^5$ etc.

Question 3

Let D, E, F be the points of contact of the circle inscribed in the triangle ABC. Suppose that the triangle DEF is similar to ABC. Prove that both ABC and DEF are equilateral.

Answer

Angles as shown on Diagram 1. $\angle FDB = y^\circ$, as angle between chord DF and tangent CDB is equal to the angle DEF in the alternate segment. We get, similarly, $\angle DFB = y^\circ$, $\angle AFE = x^\circ = \angle AEF$, $\angle CED = z^\circ = \angle CDE$ and thus, when D corresponds to A, $A = 180^\circ - 2x^\circ$, $B = 180^\circ - 2y^\circ$, $C = 180^\circ - 2z^\circ$. Similarity of the pair of triangles yields $x^\circ = 180^\circ - 2x^\circ$, $y^\circ = 180^\circ - 2y^\circ$, $z^\circ = 180^\circ - 2z^\circ$ giving $x^\circ = 60^\circ = y^\circ = z^\circ$ so that the two triangles are each equilateral. If D corresponds to C, we get (i) $x^\circ + 2z^\circ = 180^\circ$ in triangle CDE and either (ii) $2x^\circ + y^\circ = 180^\circ$ or (iii) $2x + z = 180^\circ$ in triangle AEF. (i) and (iii) give $x = z = 60^\circ$ and so $y = 60^\circ$. (ii) leads to $2y + z = 180^\circ$ in triangle FDB which, with (i) and (ii), gives $x = y = 60^\circ$.



Question 4

(i) A rectangular billiard table, with pockets only at the four corners, has dimensions 5 ft x 3 ft. A ball is hit from a corner at an angle of 45° to the sides. Prove that it will go into a pocket after several rebounds and find the number of rebounds.
It is of course assumed that the ball and pockets have conveniently small size and that the ball keeps rebounding at an angle of 45° unless it hits a pocket.

(ii) Suppose that the billiard table is 6 ft x 4 ft and the ball starts 1 ft away from a pocket on a 4 ft side. Describe the path of the ball.

(iii) If in question (i), 5 and 3 are replaced by integers p,q with no common factor, show that the ball goes into a pocket and find the number of rebounds.

(iv) Discuss the path of the ball if in question (ii), 6 and 4 are replaced by any positive integers r and s.

Answer

(i) Path as shown on Diagram 2, or:

$$(0,0) \to (3,3) \to (5,1) \to (4,0) \to (1,3) \to (0,2) \to (2,0) \to (5,3).$$

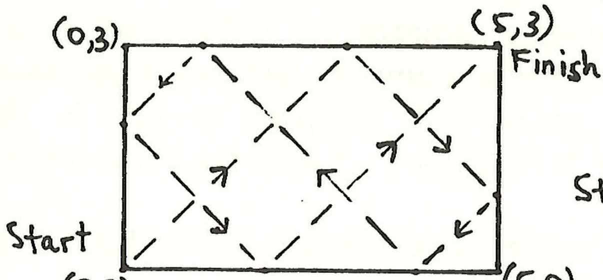


Diagram 2

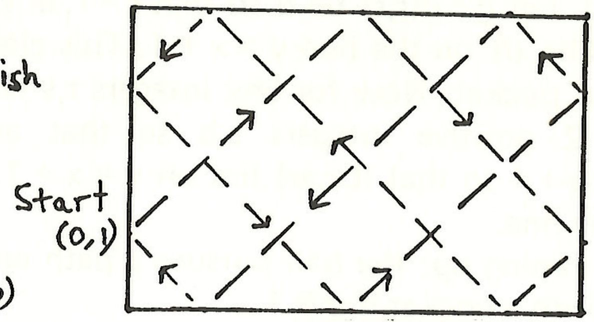


Diagram 3

(ii) Path as shown on Diagram 3 or:
$(0,1) \rightarrow (3,4) \rightarrow (6,1) \rightarrow (5,0) \rightarrow (1,4) \rightarrow (0,3) \rightarrow (3,0) \rightarrow (6,3) \rightarrow (5,4) \rightarrow (1,0) \rightarrow (0,1)$.
The above path is thus continually repeated.

(iii) The path of the ball after a rebound may be discussed by the ingenious devices of reflecting the table rather than the ball. To explain this cryptic remark consider Diagram 4. ABCD represents the table, and the path of a ball. A′, B′ and Q′ are the "mirror images" of A, B and Q in the line CD. (i.e. Each is obtained by doubling the perpendicular from the corresponding point to CD). It is easy to see that $\Delta PCQ \equiv \Delta PCQ'$ and hence that the line XPQ′ is straight.

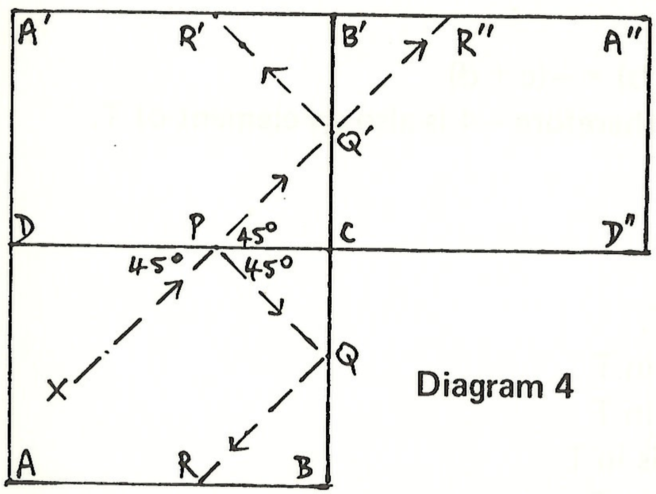


Diagram 4

Similarly, any question about the path QR after the next rebound can be answered by producing the straight line PQ′ to R″ having "reflected" the new table A′B′CD about the reflecting cushion B′C, and so on. If one constructs the "lattice" consisting of all possible reflections of the table, and reproduces the initial straight line path of the ball, one can thus completely describe the actual path of the ball. For example, it will go into a pocket if and only if the straight line passes through a corner of the lattice. So, in (i) the ball goes into a pocket because the point (15,15) lies on the line $y = x$. In (ii), the line is $y = x + 1$ so that we never have both x and y even simultaneously, and the line never passes through a point (6h,4k).

In this part of the question where the dimensions are p,q, the path is $y = x$ and the point (pq,pq) is the first point of the form (hp,kq) encountered. The number of moves taken to get there equals the number of $p \times q$ rectangles crossed and this is $(p + q - 1)$; there are therefore $p + q - 2$ rebounds.

(iv) Let g.c.d(r,s) be d so that $r = r_1 d$, $s = s_1 d$; we are to find if there is a point $(kr_1 d, hs_1 d)$ on the line $y = x + 1$. This clearly demands $d = 1$ if the ball is to go into a pocket. Now for any integers r,s with no common factors we can always find 2 positive integers a,b so that $ar - bs = \pm 1$. Suppose $ar - bs = 1$. Then $ar = bs + 1$ so that (bs,ar) lies on $y = x + 1$. If $ar - bs = -1$ then the point (ar,bs) is on the line.

Summing up: the ball pursues a path endlessly repeated as in (ii), if $d > 1$, but goes into a pocket if $d = 1$.

Question 5

Let T be a set of integers with the following properties:–

(i) T contains 0,1,3,4,5.

(ii) If a,b,c,d are distinct elements of T such that:–

$$a + b = c + d$$

then

$$r = -(a + b) = -(c + d)$$

is also in T. For example $0 + 4 = 1 + 3$; therefore -4 is also an element of T.

Prove that T contains all integers.

Answer

Taking $a = 0$, $c = 1$, we get -4, -5 in T.

Again $1 = (0 + 1) = (-4 + 5)$ so -1 is in T

$3 = (0 + 3) = (4 + -1)$ so -3 is in T

$2 = (-1 + 3) = (-3 + 5)$ so -2 is in T

and $-2 = (-2 + 0) = (-5 + 3)$ so 2 is in T.

Now suppose all the integers from $-n$ to $+n$, where $n \geqslant 5$ are in T. Then $n + 1 = (n) + (1) = (n - 1) + (2)$ so $-(n + 1)$ is in T. $-(n + 1) = (-n) + (-1) = (-n + 1) + (-2)$ so $n + 1$ is in T.

As all the integers from -5 to $+5$ are in T, and $(n + 1)$ and $-(n + 1)$ are in T if all the integers from $-n$ to $+n$ are in T, we have proved that all integers are in T.

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SOLUTIONS TO THE SCHOOL MATHEMATICS COMPETITION 1972

SENIOR DIVISION

Question 1

(i) Find an integer value of x for which $x^2 + x + 41$ is not a prime number.

(ii) If a is any given positive integer, find an integer value of x for which $ax^2 + 1$ is not a prime number.

(iii) Prove that in both cases there are infinitely many such x.

Answer

(i) An obvious solution is $x = 41$; however $x = 40$ is the smallest positive solution.

(ii) If $a \neq 1$ then $x = a$ is a solution for

$$a^3 + 1 = (a + 1)(a^2 - a + 1).$$

If $a = 1$ then $x = 3$ is a solution.

(iii) $41k$ is also a solution to (i) for any positive integer k. If we assume $a \neq 1$ then k^3a is also a solution to (ii) for any positive integer k. In case $a = 1$ then of course any odd integer greater or equal to 3 is a solution.

Question 2

Given n distinct positive integers $a_1, a_2, \ldots, a_n$, none of which is divisible by a prime number greater than 3, prove that:—

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < 3.$$

Answer

Since no integer a_i, $i = 1, \ldots, n$ is divisible by a prime number greater than 3, the factorization theorem tells us each a_i may be expressed in the form $2^{p_i}3^{q_i}$ where no pairs of indices are the same; i.e., if $i \neq j$, $(p_i,q_i) \neq (p_j,q_j)$.

Thus

$$\sum_{i=1}^{n} \frac{1}{a_i} = \sum_{i=1}^{n} \left(\frac{1}{2^{p_i}\,3^{q_i}}\right)$$
$$\leqslant \Sigma \frac{1}{2^{p_i}} \; \Sigma \frac{1}{3^{q_i}}$$

where the first summation is taken over the distinct p_i, and the second is taken over the distinct q_i.

Therefore $\sum_{i=1}^{n} \frac{1}{a_i} < \sum_{j=0}^{\infty} \frac{1}{2^j} \sum_{i=0}^{\infty} \frac{1}{3^i} = 2.\frac{3}{2} = 3.$

where we use the fact that each infinite sum is a geometric progression.

Question 3

(i) Let $a \leqslant b \leqslant c \leqslant d \leqslant e \leqslant f \leqslant g$ be seven positive integers whose sum is 28 and whose product is 5040. Clearly the numbers 1, 2, 3, 4, 5, 6, 7 have this property. Show that they are the only such numbers.

(ii) Find eight positive integers $a \leqslant b \leqslant c \leqslant d \leqslant e \leqslant f \leqslant g \leqslant h$ other than the obvious 1, 2, 3, 4, 5, 6, 7, 8 whose sum is 36 and whose product is 40320.

Answer

We offer two proofs of part (i); the first being conceptual and the second computational.

Proof 1. The 8 prime factors of 5040 are 2,2,2,2,3,3,5,7; their sum is 26. The 7 desired factors must consist of m 1's and (7−m) numbers which are either members of the original set of eight or products of some of them. The number of times we need replace a pair of numbers x,y by their product is $8 - (7 - m) = m + 1$. We then increase the sum by $d = xy - (x + y) = (x - 1)(y - 1) - 1$. Since $x > 1$, $y > 1$, d is always $\geqslant 0$. If $d = 0$ then x,y = 2,2. If d = 1 then x,y = 3,2. If d = 2 then x,y = 4,2. If d = 3 then x,y = 3,3 or 5,2 whilst if d = 4 then x,y = 2,6.

Since $d \geqslant 0$, the total sum is $\geqslant 26+m$, so m = 0, 1 or 2. If m = 0 we need the pair 4,2 but 4 is not present. If m = 2 we need three pairs of 2,2 but of course there are only four 2's. So m = 1 and we then need the pairs 2,2 and 3,2. This gives the numbers 1,2,3,2x2 = 4,5,2x3 = 6,7.

Proof 2. We choose to make repeated use of the following result the proof of which we leave as an exercise to the reader.

Proposition Suppose $M = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$ is the unique factorization of M into a product of powers of primes p_i, i = 1, . . ., n. Then the sum of the factors of any factorization of M is not less than

$$a_1 p_1 + a_2 p_2 + \dots + a_n p_n.$$

Proof. Use induction on the number $|a| = a_1 + a_2 + \dots + a_n$.

We return now to the actual problem and begin our proof by showing that the (prime) factors 7,5 are unique. To consider 7 first we suppose 21 occurs as a factor so that the product of the remaining factors is $2^4.3.5$. Then our proposition tells us that the sum of these factors cannot be less than

$4.2 + 3 + 5 = 16$ giving a total of $21 + 16 = 37 > 28$. This rules out 21 and continuing to check 14 we obtain a product of $2^3 3^2.5$ giving a minimum sum of $3.2 + 2.3 + 5 = 17$; i.e. a minimum total of 31, again a contradiction. A similar analysis reveals 5 to be unique and so we are left to show that the numbers 1,2,3,4,6 are the unique numbers with product $2^4.3^2$ and sum 16. This is an easier problem and because we have already offered one proof we leave the remaining arguments to the reader; it is not hard to show the numbers 3 and 6 occur and the remaining problem becomes almost trivial.

(ii) It is not very difficult to construct an answer by trial and error and a little foresight. We show that in fact there are three other solutions. We do this by developing the first proof of (i). The 11 prime factors of 8! are 2,2,2,2,2,2,2,3,3,5,7; their sum is 32. We replace a pair of numbers by their products $m + 3$ times, and the sum must increase by $36 - (32 + m) = 4 - m$. If $m = 0$ it is possible that the three pairs are (a) 2,6; 2,2; 2,2 (b) 3,3; 3,2; 2,2 (c) 5,2; 3,2; 2,2 (d) 4,2; 4,2; 2,2 (e) 4,2; 3,2; 3,2. We eliminate (a) and (e) because 6 and 4 are not in the original set and (d) because 4 occurs only once as 2.2. We eliminate (b) because there are only two 3's in the original set. From (c) we obtain the allowable set; 2,2,2,3,4,6,7,10. If $m = 1$ it is possible that the four pairs are (a) 3,3; 2,2; 2,2; 2,2 (b) 5,2; 2,2; 2,2; 2,2 (c) 4,2; 3,2; 2,2; 2,2 (d) 3,2; 3,2; 3,2; 2,2. We eliminate (d) because there are not enough 3's so we have the sets 1,2,4,4,4,5,7,9; 1,3,3,4,4,4,7,10 and 1,2,3,4,5,6,7,8. If $m \geqslant 2$ we can only use the pair 2,2 three times so the only possibility is $m = 2$ and then 3,2; 3,2; 2,2; 2,2; 2,2. But 2 is still used too often. We thus have four distinct solutions:— 2,2,2,3,4,6,7,10; 1,2,4,4,4,5,7,9; 1,3,3,4,4,4,7,10 and 1,2,3,4,5,6,7,8.

Question 4

(i) Prove that any convex polygon with n sides (where $n \geqslant 3$) can be dissected into $3(n - 2)$ cyclic quadrilaterals. (The opposite angles of a cyclic quadrilateral add up to 180°.)

(ii) Prove that every convex quadrilateral ABCD can be dissected into four cyclic quadrilaterals by showing that:

(a) if $\angle BAD \geqslant \angle BCD$, there exists a point X on the diagonal AC such that

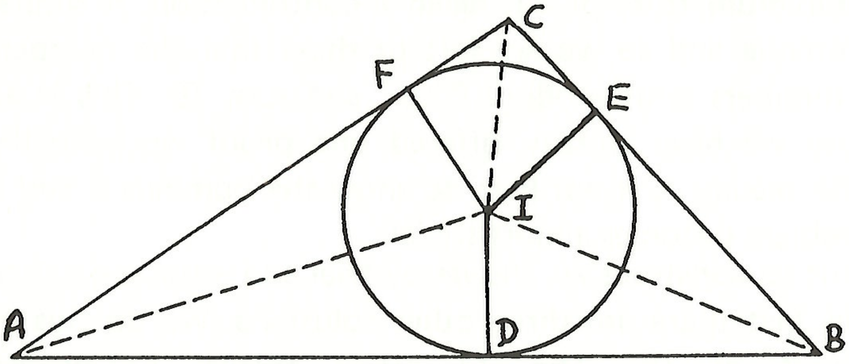
$$\angle BXD = 180^\circ - (\angle BAD - \angle BCD).$$

(b) if R is the point on AD such that $\angle DXR = \angle DCX$, there are points S, T, U on AB, BC, CD respectively such that RXSA, SXTB, TXUC and UXRD are all cyclic quadrilaterals. (A polygon is called convex if all its interior angles are less than 180°.)

(iii) Discuss which quadrilaterals can be dissected into two cyclic quadrilaterals.

Answer

(i) Suppose n = 3 so that we are considering a triangle ABC.

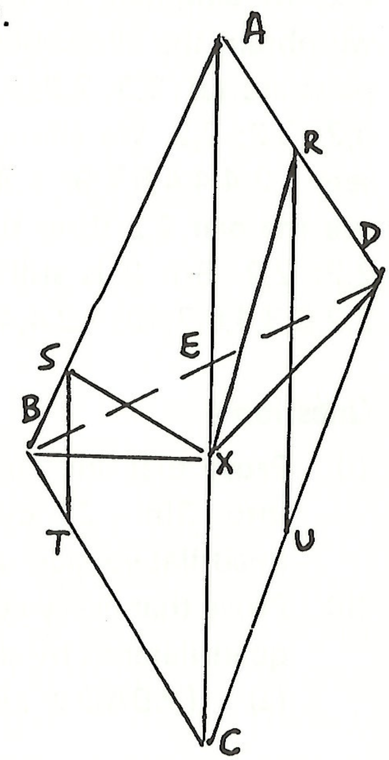


We find the incentre I and in turn the tangent points D, E, F of the incircle. The resulting quadrilaterals ADIF, BEID and CFIE are cyclic because each contains opposite angles of 90°. This establishes the result for n = 3.
Now any n-sided polygon (n > 3) can be divided into (n–2) triangles (by taking any vertex and drawing in the (n–3) diagonals). Hence an n-sided polygon can be dissected into 3(n–2) cyclic quadrilaterals.

(ii) (a) Let X belong to the interval EC. Then as X varies from E to C, ∠BXD varies from a maximum of 180° to a minimum of ∠BCD.
Now

$$180° - (\angle BAD - \angle BCD) > \angle BCD$$

since 180°–∠BAD > 0° (recall that the quadrilateral is convex).
This implies we can find X with the stated properties.

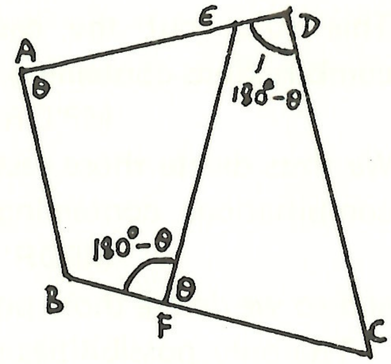


(b) Let ∠ACD = θ. Then draw RU parallel to AC. It follows that ∠RUD = ∠ACD = ∠RXD = θ and hence XUDR is cyclic. Now to find S we set ∠BXS = ∠BCA = φ say. We then construct T by drawing in ST parallel to AC. This time we have ∠BTS = ∠BCA = ∠BXS = φ and it follows that BTXS is cyclic. To check that ASXR is cyclic we note that ∠SXR = ∠BXD–($\theta+\varphi$) = 180°–(∠BAD–∠BCD)–∠BCD = 180°–∠BAD.
Finally we observe that CUXT is cyclic by noting that ∠XUD = ∠XRA = ∠BSX = ∠XTC.

(iii) The additional vertices E,F must lie on opposite sides of the quadrilateral ABCD. It follows that

$$\angle BAD = 180° - \angle ADC$$

and hence that AB is parallel to DC. This implies that the quadrilateral is a trapezium. On the other hand not every trapezium can be so dissected. We leave it as an exercise to the reader to convince himself that a necessary and sufficient condition is that ∠CBD < ∠BAD.



Question 5

A cricket team of 11 players has been selected and the following observations are made:—

(i) All the married players are either (a) professional and over thirty, or (b) under thirty with an annual income of $10,000. The unmarried players are either amateurs earning $10,000 per annum, or under thirty with an income of $6,000 per annum.

(ii) There is no player on $10,000 per annum who is not either married and amateur, or professional and over thirty.

(iii) If a player is professional and over thirty, or if he is amateur and under thirty, then either he is a married man on $6,000 per annum or a single man on $10,000 per annum.

How many amateur players are there in the team? What is the combined annual earning capacity of the team? Give reasons for your answer.

Answer

Suppose we let M, P, O and R denote the properties of being married, professional, over thirty and earning $10,000 per annum respectively and then denote the corresponding properties of being single, amateur, under thirty and earning $6,000 per annum by M′, P′, O′, R′ respectively. Thus, for example, MPOR denotes the state of being rich, old, married and professional.

We can now list the sixteen combinations of these properties by:—

MPOR	(iii)	MP′OR	(i)	MPOR′		M′P′OR′	(i)
MPO′R	(ii)	MP′O′R	(iii)	MPO′R′	(i)	MP′O′R′	(i)
M′POR	(i)	M′P′OR	(ii)	MPOR′	(i)	M′P′OR′	(i)
M′PO′R	(i)	M′P′O′R	(ii)	M′PO′R′		M′P′O′R′	(iii)

We then proceed to delete the combinations that cannot occur. To begin with condition (i) gives the only combinations containing M or M′ as

MPOR, MPOR′, MPO′R, MP′O′R

and M′P′OR, M′P′O′R, M′PO′R′, M′P′O′R′.

This rules out the cases above marked by (i). Then (ii) gives the only combinations containing R as

MP'OR, MP'O'R, MPOR, M'POR.

We thus delete those cases above marked by (ii). Finally (iii) implies that the only combinations containing PO or P'O' are

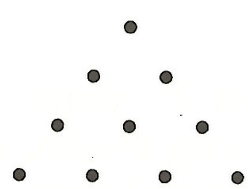
MPOR', M'POR, MP'O'R', M'P'O'R

and so we delete those possibilities marked (iii).

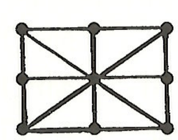
The only possibilities still remaining are MPOR' and MPO'R'. Hence it follows that all players are professionals earning $6,000 per annum. Thus there are no amateurs in the team and the team's combined income is $66,000. Of course assuming that the problem were solvable the team had to be a team of professionals or a team of amateurs with a combined income of either $66,000 or $110,000.

The following two problems were submitted by Dorothy Fermat.

Two Dotty Problems



Move 3 dots to new places, so that the above equilateral triangle appears to have rotated through an angle.



In the above square there are 8 lines containing exactly 3 dots. Move 2 dots to new places so that the resulting figure (*not* a quadrilateral) has 10 lines each containing exactly 3 dots.

(Answers on page 36)