PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class and school. Model solutions and the names of those who submit solutions by August 31, 1972 will be published in the next issue Vol 8 No 3, 1972. However the names of authors of successful solutions received between September 1, 1972 and the actual date of publication of Vol 8 No 3 will be published as late solutions in Vol 9 No 1. Send all solutions to the Editor (address on inside front cover).

Only those students who have not yet commenced their fourth year of secondary education are eligible to submit solutions of problems in the Junior Section. All students may submit solutions of problems in the Open Section.

Junior

J181 Show that the product of 4 consecutive integers is always one less than a perfect square.

J182 Find the integral solutions of the equation $y^3 - x^3 = 91$.

J183 Four equal spheres are placed on a horizontal plane so that each touches two others. A fifth equal sphere, rests on top of the other four. If the radius of a sphere is r, find the height of the highest point of the fifth sphere above the horizontal plane.

Open

O184 Let a, b, c and d be any four positive integers. Let a_1, b_1, c_1 and d_1 be the differences between a and b, b and c, c and d, and d and a respectively. The same process is then used to obtain a_2, b_2, c_2 and d_2 (e.g. $a_2 = |a_1 - b_1|$) and so on. Show that eventually four zeros must be obtained.

For example, starting with

7 24 32 32

one obtains in succession

17	8	0	25
9	8	25	8
1	17	17	1
16	0	16	0
16	16	16	16
0	0	0	0

0185 Given the sequence of numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . . where each number (beginning with the third) is the sum of the 2 preceding numbers; show that there exists a number amongst the first 100,000,001 terms of this sequence terminating with four zeros.

0186 Solve the equation

$$\sqrt{a - \sqrt{a + x}} = x.$$

0187 (i) How many roots has the equation

$$\cos x = \frac{x}{50} ?$$

(ii) How many roots has the equation

$$\cos x + \log_e x = 0?$$

0188 The following excellent problem appeared in the 1967 Tasmanian Schools Mathematics Competition and we reprint it by courtesy of the Mathematical Association of Tasmania. It was not solved by the competitors.

A pupil is situated at the centre of a circular swimming pool. A non-swimming teacher (who wishes to administer punishment) waits for him at the edge. He can run 4 times as fast as his quarry can swim, but cannot catch him on land. Has the pupil a strategy by which he can escape from the pool and elude capture?

0189 ABCD is a cyclic quadrilateral and M, N, P, Q are the midpoints of sides CD, DA, AB, BC respectively. MS, NT, PU, QV are perpendicular to AB, BC, CD and DA respectively. Prove that they are concurrent.

O190 In a round robin soccer tournament (i.e. each team plays every other team) no games are goal-less, no two results are identical and the number of goals scored in any one game does not exceed the number of teams. Two points are awarded for a win and 1 point each for a draw. The difference of the point scores of the first two teams is equal to four times the difference of the point scores of the last two teams. No two teams emerged with the same number of points. How many teams were entered in the competition?

SOLUTIONS

Solutions to Problems 171–180 in Vol 8 No 1

The names of successful problem solvers appear after the solution to Problem 180.

JUNIOR

J171 Prove that for $n > 2$,

$$(n!)^2 > n^n.$$

Answer $(n!)^2 = (1 \times 2 \times 3 \times \ldots \times n)^2$
$= (1 \times 2 \times \ldots \times n) \times (n \times (n-1) \times \ldots \times 2 \times 1)$
$= (1 \times n) \times (2 \times (n-1)) \times (3 \times (n-2)) \times \ldots$
$\times ((n-1) \times 2) \times (n \times 1).$

This expresses $(n!)^2$ as the product of n factors each of the form $(k+1)(n-k)$ where k is one of $0,1,2,\ldots(n-1)$.

Since $(k+1)(n-k) = n + k(n-k-1)$ each of the factors is greater than n except the first and last, which are equal to n.

Hence $(n!)^2 > n^n$, when $n > 2$.

J172 There are certain pairs of integers x,y for which $2x + 3y$ is divisible by 11, e.g. $x = 1$, $y = 3$. Prove that for exactly the same pairs of integers also $7x + 5y$ is divisible by 11.

Answer 11 is a factor of $2x + 3y$
$\Leftrightarrow$ 11 is a factor of $9.(2x + 3y)$
$\Leftrightarrow$ 11 is a factor of $(18x + 27y) - 11x - 22y$
$\Leftrightarrow$ 11 is a factor of $7x + 5y$.

($\Leftrightarrow$ can be "translated" as *if and only if*, or as *implies and is implied by*.)