O190 In a round robin soccer tournament (i.e. each team plays every other team) no games are goal-less, no two results are identical and the number of goals scored in any one game does not exceed the number of teams. Two points are awarded for a win and 1 point each for a draw. The difference of the point scores of the first two teams is equal to four times the difference of the point scores of the last two teams. No two teams emerged with the same number of points. How many teams were entered in the competition?

SOLUTIONS

Solutions to Problems 171–180 in Vol 8 No 1

The names of successful problem solvers appear after the solution to Problem 180.

JUNIOR

J171 Prove that for $n > 2$,

$$(n!)^2 > n^n.$$

Answer $(n!)^2 = (1 \times 2 \times 3 \times \ldots \times n)^2$
$= (1 \times 2 \times \ldots \times n) \times (n \times (n-1) \times \ldots \times 2 \times 1)$
$= (1 \times n) \times (2 \times (n-1)) \times (3 \times (n-2)) \times \ldots$
$\times ((n-1) \times 2) \times (n \times 1)$.

This expresses $(n!)^2$ as the product of n factors each of the form $(k+1)(n-k)$ where k is one of $0, 1, 2, \ldots (n-1)$.

Since $(k+1)(n-k) = n + k(n-k-1)$ each of the factors is greater than n except the first and last, which are equal to n.

Hence $(n!)^2 > n^n$, when $n > 2$.

J172 There are certain pairs of integers x,y for which $2x + 3y$ is divisible by 11, e.g. $x = 1$, $y = 3$. Prove that for exactly the same pairs of integers also $7x + 5y$ is divisible by 11.

Answer 11 is a factor of $2x + 3y$
$\Leftrightarrow$ 11 is a factor of $9.(2x + 3y)$
$\Leftrightarrow$ 11 is a factor of $(18x + 27y) - 11x - 22y$
$\Leftrightarrow$ 11 is a factor of $7x + 5y$.

($\Leftrightarrow$ can be "translated" as *if and only if*, or as *implies and is implied by*.)

J173 The square of a number X has the digit 7 at the tens. What is the last digit of X^2?

Answer If the last digit of X is a, we have $X = 10b + a$ where a,b are integers with

$$0 \leqslant a \leqslant 9.$$

Then $X^2 = 100b^2 + 20ab + a^2 = 20(5b^2 + ab) + a^2$.

Hence, on division by 20, X^2 leaves the same remainder as a^2 and since the "tens" digit of X is 7, this remainder is known to lie between 10 and 19 inclusive. Checking the squares of the 10 digits we find that only two of them viz 4 and 6, have squares leaving remainders in this range. In both cases, the final digit of a^2 and hence also of X^2 is 6.

OPEN

O174 How many n-digit numbers can one form from the digits 1,2,3 alone, so that each of the three digits should appear at least once in the number?

Answer Let N_k be the number of n-digit numbers constructible from k digits, each appearing at least once. We are asked to find N_3. Notice that $N_1 = 1$. Also $N_2 = 2^n - 2N_1$ since there are 2 choices of digit for each place, altogether 2^n numbers can be constructed, but these include the 2 numbers containing only one digit. Similarly if 3 digits are available, altogether there are 3^n ways of filling the n places with these digits. But this includes all the 2-digit numbers using digits 1,2 or 1,3 or 2,3 and the 3 one digit numbers

i.e.
$$N_3 = 3^n - 3.N_2 - 3N_1$$
$$= 3^n - 3(2^n - 2) - 3.$$

This is the required answer.

To generalise a little, one can obtain the formula

$$N_k = k^n - {}^kC_{k-1}N_{k-1} - {}^kC_{k-2}N_{k-2} - \ldots - {}^kC_1N_1$$

from which the expressions N_k can be calculated in succession, or, better still, one can prove that $N_k = \sum_{r=1}^{k} {}^kC_r(-1)^{k-r}r^n$ and so calculate N_k directly. (Would any reader care to submit a proof of this for publication? — Problem Editor)

O175 Let $a^2 + b^2 + c^2 = 1$. Prove that

$$-\tfrac{1}{2} \leqslant ab + bc + ca \leqslant 1.$$

Answer $0 \leqslant (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

$$\therefore \quad -\tfrac{1}{2} = -\frac{(a^2+b^2+c^2)}{2} \leqslant ab + ac + bc.$$

Also $0 \leqslant (a - b)^2 + (b - c)^2 + (c - a)^2 = 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca.$

$$\therefore \quad ab + bc + ca \leqslant a^2 + b^2 + c^2 = 1.$$

0176 Prove that if we are given any n positive integers $a_1, a_2, \ldots, a_n$ there are always integers k,m, $1 \leqslant k \leqslant m \leqslant n$ so that

$$a_k + a_{k+1} + \ldots + a_{m-1} + a_m \text{ is divisible by n.}$$

e.g. given 13, 5, 18, we have $13 + 5$ ($= a_1 + a_2$) divisible by 3; we also have $18 = a_3$ divisible by 3, so that in the last example $k = 3 = m$.

Answer Let $S_t = a_1 + a_2 + \ldots + a_t$ for $t = 1,2, \ldots n$, and let $S_t = q_t n + r_t$ where q_t and r_t are integers, and $0 \leqslant r_t < n$. [That is, q_t and r_t are respectively the quotient and remainder when the positive integer S_t is divided by n.] Since there are only n different possible remainders $(0,1,2, \ldots, n-1)$ and there are n integers S_t, either the remainder 0 occurs for some t (S_p say), or alternatively two of the S_t leave the same remainder (S_u and S_v, say). In the first case $a_1 + a_2 + \ldots + a_p$ is divisible by n; in the second case

$$(q_v - q_u)n = S_v - S_u = a_{u+1} + a_{u+2} + \ldots + a_v$$

is divisible by n.

0177 *(i)* D, E, F are points on the line segments BC, CA, AB such that the circles through CDE, BDF intersect again at the point P in the interior of the triangle ABC. Prove that the circle through A, E, F also passes through P.

(ii) If D lies on BC produced so that DEF is a straight line prove that P also lies on the circumcircle of the triangle ABC.

Answer *(i)* Whether P is inside ABC as stated (see Fig. 1), or outside (Fig. 2) the

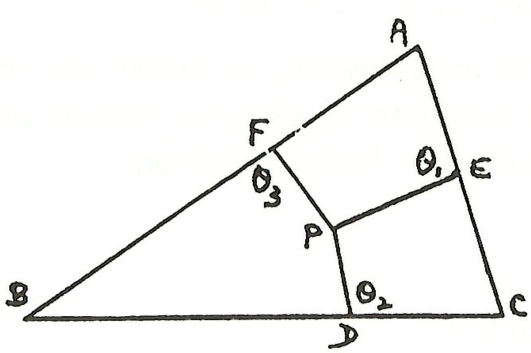


Fig. 1

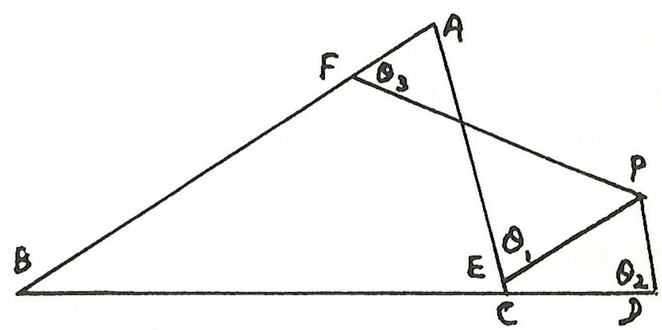


Fig. 2

four points A,F,P,E are concyclic since
$\theta_1 = \theta_2$ (exterior angle of cyclic quadrilateral = interior opposite angle).
$= \theta_3$ (same reason).

In Fig. 1, AFPE is a cyclic quadrilateral since one pair of opposite angles is now seen to be supplementary.

In Fig. 2, the chord AP subtends equal angles at F and E, whence the four points are concvclic.

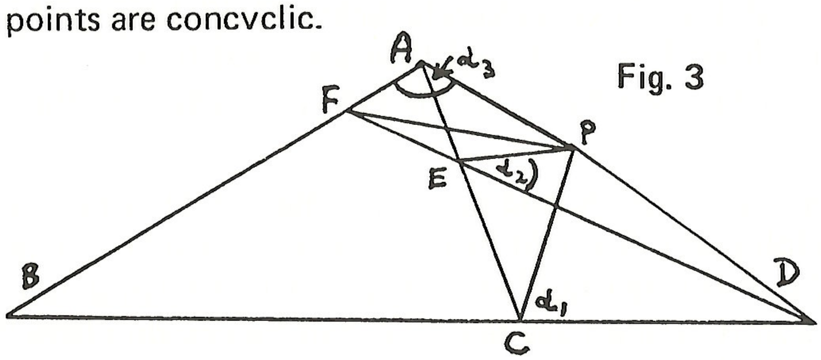
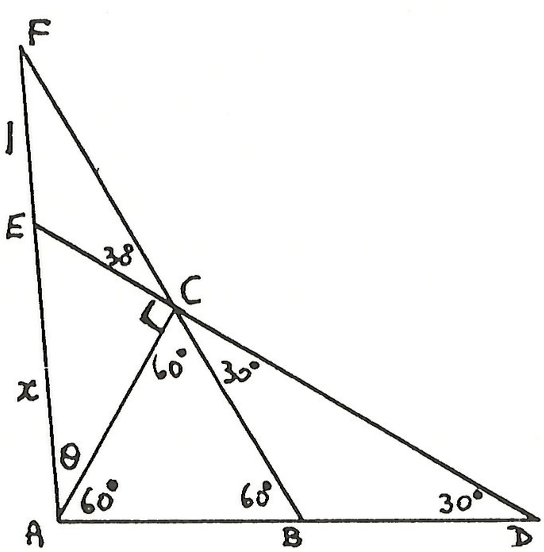


Fig. 3

(ii) Fig. 3 now differs from Fig. 2 only in that DEF is straight. We have $\alpha_1 = \alpha_2$ (since DCEP is cyclic) $= \alpha_3$ (since AFEP is cyclic) Therefore CPAB is cyclic (as α_1 and α_3 are an exterior angle and the interior opposite angle of this quadrilateral).

0178 Prove the correctness of the construction given for obtaining $^3\sqrt{2}$ on page 10 last issue (Vol 8 No 1).

Answer Let $\angle CAE = \theta$. It is easy to see that $\angle CFE = 60^\circ - \theta$ after verifying the angles marked on the figure. From the right angled triangle ACE with *AE = x units and *AC = 1 unit we have $\cos\theta = \frac{1}{x}$, $\sin\theta = \frac{\sqrt{(x^2-1)}}{x}$.



Applying the sine rule to triangle ACF gives

$$\frac{\sin(60^\circ - \theta)}{1} = \frac{\sin(90^\circ + 30^\circ)}{(1+x)} \quad \text{so that}$$

$$(1+x)(\sin 60^\circ \cos\theta - \cos 60^\circ \sin\theta) = \sin 120^\circ$$

$$(1+x)\left(\frac{\sqrt{3}}{2}.\frac{1}{x} - \frac{1}{2}.\frac{\sqrt{(x^2-1)}}{x}\right) = \frac{\sqrt{3}}{2}$$

which simplifies to $(1+x)\sqrt{(x^2-1)} = \sqrt{3}$ and finally to $x^4 + 2x^3 - 2x - 4 = 0$.

The L.H.S. factorises into $(x^3-2)(x+2)$ whence the solutions are $x = {}^3\sqrt{2}$ or $x = -2$. Since the negative root is clearly inadmissible we have $x = {}^3\sqrt{2}$ as required.

O179 This is the promised puzzle due to Raymond Smullyan. Three men A, B and C are all known to be perfect logicians; that is, when presented with any body of premises all the valid consequences are instantaneously clear to them.

They are shown a set of 8 markers, 4 black and 4 white, and then blindfolded. Two markers are attached to the forehead of each man and the remaining two are concealed before their blindfolds are removed.

A is asked if he knows the colours of the markers he bears. "No", he says. The same answer is elicited in turn from B, from C and then a second time from A. Now B announces that he knows the colours of his markers. What are they? Explain.

Answer B announces that he has one white and one black marker. He rules out the possibility of 2 blacks as follows:— A would then have known that he did not have 2 black markers, since C in that case would immediately have known that his markers must be white. But also A would have reasoned that he couldn't have 2 white markers, since in that case C could have worked out that his markers must be one of each colour by the prior indecision of A and B. Hence, if B had 2 black markers, A would have been able to deduce that he must have one of each.

By an identical argument, B rules out the possibility of having 2 white markers.

O180 Joe brought \$2500 with him to a two-up club and employed the following system of betting. He started with one dollar and trebled the bet after each loss and started again with \$1 after a win. All the money he won plus the capital in that win, he put into a bag. After having won his last bet and gone through the \$2500 exactly, he went to celebrate and mislaid the bag which was found by an honest club manager. The manager then announced that he would give the bag to the person who knew the exact amount in the bag because there were several claimants. All Joe could remember was that he had won exactly 100 bets but after consulting a mathematician-friend, the amount was identified and the bag returned to Joe. How much was there in the bag? (This problem was sent in by Mr J. Tong, an ex-school Parabola reader. — C.D. Cox.)

Answer Suppose Joe won after n bets preceded by (n—1) losses in a particular cycle.

Then amount lost $L = 3^{n-1} + 3^{n-2} + \ldots + 3 + 1$

and outlay $= 3^n + 3^{n-1} + \ldots + 3 + 1.$

Amount into bag $= 2.3^n$.

Noting that $3^n - 1 = 2(3^{n-1} + 3^{n-2} + \ldots + 3 + 1)$, we then have that

the outlay $= 3L + 1$, and amount

into bag $= 4L + 2$.

Since Joe had 100 wins and the total outlay was \$2500 then $3(L_1 + L_2 + \ldots + L_{100}) + 100 = 2500$. This gives $L_1 + \ldots + L_{100} = 800$.

Amount into bag $= 4(L_1 + L_2 + \ldots + L_{100}) + 200$
$= \$3400$.

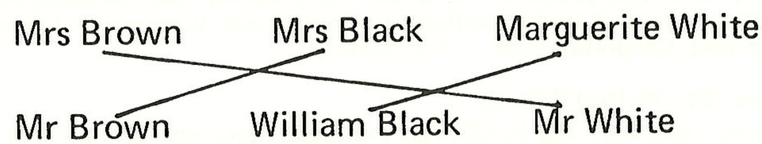
As there were some omissions from the solution to Problem O170 published on pages 38–39 of our last issue, Vol 8 No 1, the correct solution is printed again.

O170 This puzzle in pure logic is due to Raymond Smullyan of Princeton University. On Christmas Day 1970, three married couples celebrated by dining together. (1) Each husband was the brother of one of the wives; that is, there were three brother-sister pairs in the group. (2) Helen was exactly 26 weeks older than her husband who was born in August. (3) Mr White's sister was married to Helen's brother's brother-in-law. (4) She (Mr White's sister) married him on her birthday which was in January. (5) Marguerite White was not as tall as William Black. (6) Arthur's sister was prettier than Beatrice. (7) John was fifty years old.

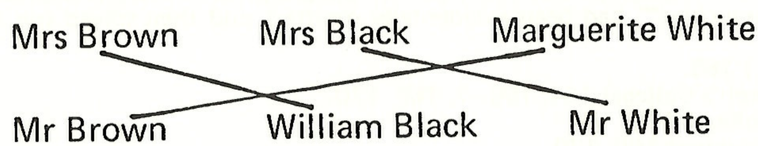
What was Mrs Brown's first name?

Answer: Using statements 1 and 5 only we see that the people present are represented by one or other of the two diagrams A and B

A



B



where the lines indicate brother-sister relationships.

Let us consider A first. There are two ways of assigning the first names, Arthur and John to the surnames Brown and White.

Case A_1 *Arthur White and John Brown.* Using 6 we see that Mrs Brown is not Beatrice so the women are Helen Brown and Beatrice Black. This situation does not conflict with statement 3, but it does conflict with 2, 4 and 7. Even if Helen's birthday was on January 31, 1920 and John's on August 1, she is 26 weeks + 1 day older than he is (remembering that 1920 was a Leap Year).

Case A_2 *Arthur Brown and John White.* Using 6, the women are Beatrice Brown and Helen Black. This is in conflict with 3 since Arthur Brown is not his own brother-in-law.

Case B_1 *Arthur White and John Brown.* Using 6, the women are Beatrice Brown and Helen Black. No conflict is found with any statement.

Case B_2 *Arthur Brown and John White.* If the women are Helen Brown and Beatrice Black, statement 3 is not true. There the women's names must be Beatrice Brown and Helen Black.

Hence in both of the allowable solutions Mrs Brown's first name is Beatrice.

Solvers of Problems 171–80 (Vol 8 No 1)

M. Buchan (Sydney Boys High) 175.
K. Burns (Campbell High) 174 (with generalisation), 175–8, 179 (good), 180 (not quite a complete argument).
P. Corkish (Belmore Boys High) 179, 180.
H. de Feraudy (St Joseph's College, Hunter's Hill) 174, 176–8, 180.
S. Hing (St Joseph's College) 174–5, 178, 180.
G. Holmila (Peakhurst High) 175, 177–8.
J. Holten (East Hills Boys High) 173, 179 (argument only partly correct).
R. Hopkins (Sydney Boys High) 174–8, 179 (argument only partly correct), 180.
R. Kuhn (Sydney Grammar) 174–6, 179 (very clear argument).
D. MacLeod (Beacon Hill High) 174–6, 178.
J. Maguire (Greystanes High) 174–80.
A. Martin (St Joseph's College) 180.
D. Paterson (South Sydney Boys High) 172.
B. Quinn (Marist Brothers, Hamilton) 174–5, 177–8, 179 (argument only partly correct), 180.
M. Reid (St Joseph's College) 175, 177–8.
A. Szeto (St Joseph's College) 175 (proved a more general result), 177–8, 180.
P. Wild (Hollywood Senior High School, Cottesloe, WA) 171–3.

[I myself found O180 quite a hard problem. For once, Keith Burns, who has submitted many very brilliant answers, has revealed glimmerings of human failings. However I was surprised and pleased by the number of correct arguments submitted. Congratulations. – C.D. Cox.]

Late Solvers of Problems 161–70 (Vol 7 No 3)

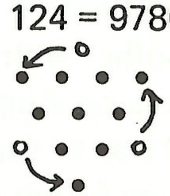
K. Burns (Campbell High) 164, 166, 168 (O168 was the faulty question. Keith Burns cleverly deduced that the additional condition, "For every pair of people present there was exactly one club of which both were members", had been accidentally omitted, and then solved the correct question), 169 (good), 170.
J. Cranney (S.C.E.G.S.) 168.
H. de Feraudy (St Joseph's College) 164, 166–7, 169, 170.
S. Hing (St Joseph's College) 166.
A. Pollack (Sydney Grammar) 166, 170.
M. Temperly (Campbell High) 164, 165 (partly), 170.

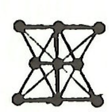
☆ ☆ ☆

Answers

"Seven": 12,128,316 ÷ 124 = 97809

Two Dotty Problems:





Consecutive Digits: 13,452; 13 x 4 = 52

Figure the Matches: 36 matches