THE PENDULUM

1. The physical system and the forces acting on it

The system which we consider consists of a pendulum bob of mass m connected by a rigid rod of negligible mass to the pivot. This system is illustrated in Figure 1. We emphasise that we are talking about a rigid rod rather than a flexible string, so that it is possible for the pendulum to swing all the way up into vertical position on top of the pivot and stay there for an appreciable time.

At any given moment the pendulum rod makes an angle θ with the downward vertical direction, as indicated in Figure 1. The position of stable quilibrium is $\theta = 0$, when the mass is in its lowest possible position. We shall assume that the entire system is free of friction, or more practically, that frictional forces can be ignored to a good enough approximation. The purpose of this discussion is to find out what happens when the usual approximation of only small angles of swing is no longer valid.

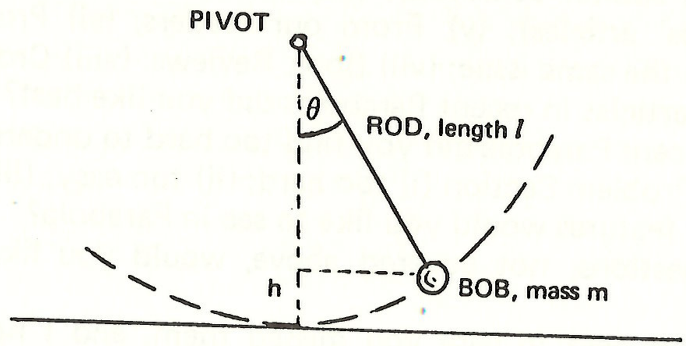

Figure 1: The pendulum with rigid, massless rod

Let us look at the forces which act on the pendulum bob in the position indicated in Figure 1. These forces are two, namely the force of gravity which acts downward and has a magnitude equal to mg, and the force exerted by the rod, which force we shall call the tension and denote by τ. This latter force acts in the direction of the rod. The two forces are shown in Figure 2. Also shown is the decomposition of the force of gravity into two components, one component in the direction along the rod away from the pivot, and the other component at right angles to it.

Since the length l of the rod is constant (assumption), it is convenient to rewrite the vector equation of motion,

$$\vec{F} = m\vec{a}, \tag{1.1}$$

by decomposing the equation into two components, one along the direction of the rod and pointing towards the pivot, and the other at right angles to that direction. Let us first consider the components at right angles to the rod. The arc length from the equilibrium point to the actual position of the pendulum bob as shown in Figure 1 is given by $l\theta$ where of course the angle θ is measured in radians. The time derivative of this gives the component of velocity in the direction of increasing θ, i.e. positive if the pendulum bob is moving upward towards the right. This velocity component is given by

$$v_\theta = \frac{d}{dt}(l\theta) = l\frac{d\theta}{dt}. \tag{1.2}$$

Finally, the acceleration component in the same direction is the time derivative of this velocity component,

$$a_\theta = \frac{d}{dt}(v_\theta) = l\frac{d^2\theta}{dt^2}. \tag{1.3}$$

This actually needs some justification, and interested readers are referred to the article on planetary motion in Parabola Vol 7 No 2.

Let us now write down the component of Newton's law, equation (1.1), in the direction of increasing θ. We know the acceleration component from equation (1.3), and we know the component of the force by looking at Figure 2, and noting that this component, of value $mg\sin\theta$, acts in such a direction as to attempt to make the angle θ smaller, and therefore must be given a negative sign.

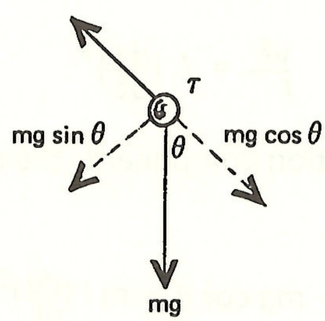

Figure 2: The forces acting on the pendulum bob

The component of equation (1.1) therefore reads as follows:

$$-mg\sin\theta = ml\frac{d^2\theta}{dt^2}. \tag{1.4}$$

The first thing to notice about equation (1.4) is that the mass of the pendulum bob cancels out altogether. This is an important fact, indicating that the motion of the pendulum is quite independent of the mass of the pendulum bob, to the extent that our basic approximation of ignoring friction is valid. In practice, if the pendulum has a higher mass, the tension and weight become bigger, and for the same speed of motion frictional forces are likely, although not certain, to be relatively less important. But if friction can be ignored, the mass of the pendulum bob does not matter at all. Cancelling out the mass, we get the equation of motion for the angle θ in the following form:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta . \tag{1.5}$$

The second important fact about this equation of motion is that we don't have to know the tension in the supporting rod at all in order to determine how the angle θ depends on time. If we know the value of θ at time $t = 0$ for example, as well as the value of the angular velocity $d\theta/dt$ at the same time then equation (1.5) can be solved, at least in principle, to give θ as a function of time for future times. Nothing else is needed.

Nonetheless it is of interest to also determine the tension τ. We do this by writing down the other component of the vector equation (1.1), namely the component corresponding to a direction along the rod pointing towards the pivot. The acceleration in this direction can be obtained from the usual formula for the centrifugal acceleration, namesly v^2/r. In this case the speed v is given by equation (1.2), the value of r is the length l so that

$$\frac{v^2}{r} = l \left(\frac{d\theta}{dt}\right)^2 .$$

With this value of the acceleration component, the component of the equation of motion (1) now becomes

$$\tau - mg\cos\theta = ml\left(\frac{d\theta}{dt}\right)^2 . \tag{1.6}$$

It is important to note that this does not mean that the tension in the supporting rod is equal to the component $mg\cos\theta$ of the weight. This is true only under very special circumstances, namely when the angular velocity $d\theta/dt$ happens to be 0. In the general case, we can solve the tension in the rod from equation (1.6) to get

$$\tau = \text{mg} \cos\theta + \text{m}\,l\,(\frac{d\theta}{dt})^2. \qquad (1.7)$$

In principle, we have now solved certain problems. The motion itself, that is θ as a function of time, is obtained from the differential equation (1.5) and the necessary initial conditions, for example at time $t = 0$ the value of θ and $d\theta/dt$ might be given. Once θ is known as a function of time we then get the tension in the rod from equation (1.7). All that remains, therefore, is to solve the differential equation (1.5). However this turns out to be far from trivial, and the solution cannot be expressed by means of the common functions of differential calculus.

2. The Energy Integral

In this, as in many other problems in mechanics, it is possible to get useful and very valuable information from the so-called energy integral. For the sake of convenience, since the mass drops out of equation (1.5) anyway, we shall be dealing with the energy per unit mass, rather than the quantity which is normally called energy in mechanics.

To obtain this energy per unit mass, we multiply both sides of equation (1.5) by the quantity $l^2 d\theta/dt$. This results in

$$l^2 \frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + lg\sin\theta \,.\, \frac{d\theta}{dt} = 0.$$

This equation can also be written in the form

$$\frac{d}{dt}\,[\tfrac{1}{2}\,(l\frac{d\theta}{dt})^2 + lg\,(1-\cos\theta)] = 0, \qquad (2.1)$$

therefore the quantity in square brackets does not change with time. This quantity is the energy per unit mass and we shall denote it as E,

$$E = \tfrac{1}{2}\,(l\frac{d\theta}{dt})^2 + lg\,(1-\cos\theta). \qquad (2.2)$$

The first term is $\frac{1}{2}v^2$, the kinetic energy divided by the mass m. The second term is of the form mgh, again divided by the mass m, i.e. of the form gh where

$$h = l(1 - \cos\theta)$$

is the height above the equilibrium position of the pendulum bob, as seen from the geometry of Figure 1. This second term is the potential energy per unit mass.

In order to determine the value of the energy E, we merely need to know its value at any one time, since we know from equation (2.1) that the value of E does not change with time. To be definite, we start off the pendulum by giving it an initial angular velocity of value ω_0 in the positive direction, starting at its equilibrium position. These initial conditions are given below:

$$\text{At time } t = 0: \ \theta = 0, \frac{d\theta}{dt} = \omega_0. \tag{2.3}$$

When we substitute these particular initial conditions into the definition (2.2) of the energy, we obtain the value

$$E = \tfrac{1}{2} l^2 \omega_0^2 \tag{2.4}$$

We shall use the notation

$$T = \tfrac{1}{2} l^2 \left(\frac{d\theta}{dt}\right)^2 \tag{2.5}$$

for the kinetic energy, and

$$V = l.g.(1 - \cos\theta) \tag{2.6}$$

for the potential energy, so that the constancy of the energy takes the following form:

$$T + V = E = \tfrac{1}{2} l^2 \omega_0^2. \tag{2.7}$$

This equation is called the energy integral.

3. Use of the Energy Integral to discuss the motion

We shall now use these results to give a qualitative discussion of the possible motions which this system of Figure 1 can perform. The only thing additional to equations (2.5), (2.6) and (2.7) which we require is the obvious fact that the kinetic energy, equation (2.5), can never become negative. Its smallest possible value is 0. Therefore when we rewrite equation (2.7) in the form $E - V = T$, we obtain the inequality $E - V \geqslant 0$, and therefore referring each to equation (2.4),

$$V \leqslant E = \tfrac{1}{2} l^2 \omega_0^2. \tag{3.1}$$

It is therefore of obvious interest to plot the behaviour of the function V, the potential energy function, as a function of the position angle θ of the pendulum. This is done in Figure 3. We see that the lowest possible value of the potential energy is 0, and the highest possible value is given by

$$V_{max} = 2\,l\,g \qquad (3.2)$$

We can now distinguish four quite separate cases, namely:

Case 1. E = 0. In this case the pendulum hangs at rest in its equilibrium position and there is no motion at all. That is, in equation (2.7) the kinetic energy T is exactly 0 at all times, and V is at its minimal value 0.

Case 2. This case is defined by the inequality

$$0 < E < 2\,l g \qquad (3.3)$$

and a typical example of an energy value in this range is shown in Figure 3. If we draw a dotted horizontal line at this value of the energy, then this line meets the

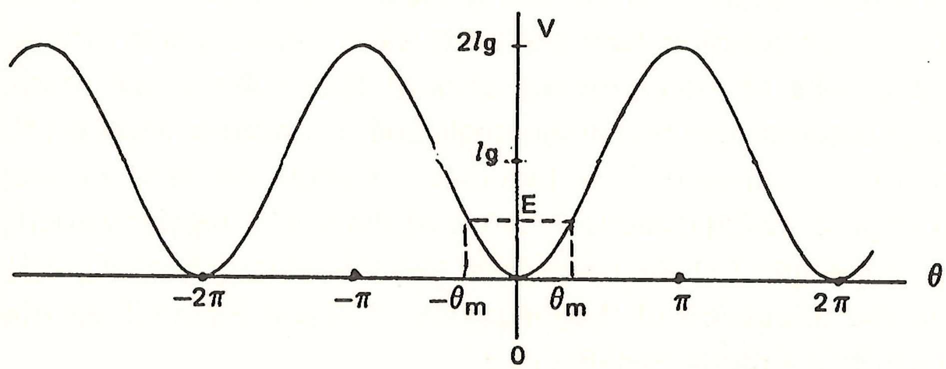

Figure 3: The potential energy (per unit mass) V plotted vs. the position angle θ of the pendulum

potential energy curve at the two abscissae indicated on the figure as θ_m and $-\theta_m$, respectively. These two values of the angle θ define the maximum extent of the possible motion of the pendulum around its equilibrium position, if the given energy (equation (2.4)) falls within the range (3.3). The motion of the pendulum is the typical to and fro pendulum-like motion between these two extreme positions. However, unless the angle θ_m turns out to be small compared to one radian, the motion is by no means a simple harmonic function.

Let us find the value of the maximum angle of swing θ_m from the initial conditions (2.3) which lead to the energy value (2.4). At the extreme point of the motion, when the pendulum bob is just turning around, the angular velocity $d\theta/dt$ is 0 at this moment, and hence so is the kinetic energy (2.5). The energy integral, equation (2.7), therefore takes the form V = E at the extreme points of swing. Substituting equation (2.4) for E and equation (2.6) for V, we obtain

$$l.g.(1 - \cos\theta_m) = \tfrac{1}{2} l^2 \omega_0^2 \tag{3.4}$$

We now use the trigonometric identity $1 - \cos x = 2 \sin^2 (\frac{x}{2})$ to obtain

$$\sin^2 (\frac{\theta_m}{2}) = \frac{l\omega_0^2}{4g} ,$$

and using the inverse sine function,

$$\theta_m = 2 \sin^{-1} (\frac{\omega_0}{2} \sqrt{(1/g)}). \tag{3.5}$$

The inequality (3.3) ensures that the argument of the inverse sine in equation (3.5) is between 0 and unity, as it must be.

Note that the result (3.5) is by no means trivial or what one would guess at by some method of rough guessing. Nonetheless it is a completely scientific result for our problem and tells us an enormous amount about the actual motion. The pendulum swings between this maximum angle and its opposite angle on the other side, going to and fro between these two extreme positions. And we now know the maximum angle of swing directly in terms of the initial angular velocity of the pendulum, ω_0, the length l of the pendulum, and the acceleration of gravity g.

Returning to the discussion of the inequality (3.3) and Figure 3, we now reach the third case which is a rather peculiar one.

Case 3. Energy equals the maximum possible potential energy. In terms of an equation, this condition reads

$$E = V_{max}. \tag{3.6}$$

Substituting the value (2.4) for the energy and the value (3.2) for the maximum potential energy we obtain the following value for the initial angular velocity ω_0 for this very special case:

$$\omega_0 = 2\sqrt{(l/g)} . \tag{3.7}$$

There is another way of obtaining the same value of energy. Instead of starting with the initial conditions as given by equation (2.3) we start off the pendulum at rest, but in its position of unstable equilibrium, with the pendulum bob direct vertically above the pivot. This is an equilibrium position, in the sense that when things are carefully enough balanced, there is no tendency for the pendulum bob to move downwards either to the right or to the left. But it is of course an unstable equilibrium. The slightest perturbing force is enough to move the pendulum away from this position and make it fall down on one or the other side. Nonetheless the "motion" in this position of unstable equilibrium must be recognised as a possible motion of the system, namely, no motion at all.

Case 4. Energy larger than the maximum potential energy. The inequality which defines this case is

$$E > V_{max}. \tag{3.8}$$

When we substitute from equations (2.4) and (3.2), we find that this condition amounts to the inequality

$$\omega_0{}^2 > \frac{4g}{l}. \tag{3.9}$$

The qualitative nature of the motion in Case 4 is completely different from the motion in any of the earlier cases. We can never get into a situation where the energy is exactly equal to the potential energy; this means that the kinetic energy can never be zero. Rather, no matter what the angle θ might be, the energy E in equation (2.7) is larger than the potential energy T. Thus the pendulum does not ever stop, but merely keeps on rotating about the pivot with varying speed. The speed of rotation is highest when the pendulum is near its bottom position, and smallest when the pendulum is near its top position directly above the pivot. But at no time is the speed of motion reduced to zero, and the nature of the motion is therefore completely different from the one which we normally think of as "pendulum motion." Nonetheless this is a valid and quite permissible and observable motion of a pendulum.

Note also that inequality (3.9) is a condition on the square of the initial angular velocity ω_0 with which we start off the pendulum. This allows either a positive or a negative value of ω_0. If the initial angular velocity ω_0 is positive, meaning that we start off the pendulum moving in counter clockwise direction, then it continues moving in a counter clockwise direction at all future times. If the initial angular velocity ω_0 is negative, then the pendulum continues to rotate in the clockwise direction at all times. If there is a high enough initial angular velocity to satisfy the condition (3.9) for Case 4, the pendulum never changes its direction of motion.

It is of extreme importance to notice how much information one can obtain about the qualitative nature of the motion by using the energy integral, without at any stage really solving the equation of motion (1.5). The actual solution of that equation involves mathematics well beyond that which is available to high school students. Nonetheless a great deal of essential information can be obtained, including a definite expression for the period of the motion in all four cases, by making use of the energy integral. We may come back to this discussion on another occasion.

J. Blatt

This is an abridged version of a talk given by Professor Blatt of the School of Mathematics, University of New South Wales, to a Sydney wide gathering of First Level Mathematics students, held on 21.5.72 at the University of Sydney.

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