THE THEORY OF GAMES

There is a fundamental difference between games of chance, such as roulette or two-up, and games of skill, such as poker or chess, which is shown by their description. Games of chance can be completely analysed by the elementary theory of probability so as to tell us whether a particular game is fair or whether it favours a certain player. It is not always realised, however, that games of skill can also be analysed mathematically; indeed an extensive mathematical theory exists which will prescribe a best method of playing many games of skill.

To apply mathematics to a game of chance or a game of skill we must start by thinking ourselves into the role of a particular player, set down all the possible outcomes of the game and then assign to each such outcome a number which measures the worth of the outcome to the player we are considering. Thus if one outcome means a gain of \$3 to the player we assign the number 3 to it, while if another outcome brings him a loss of \$2, we assign the number −2 to it. In other cases when no betting is involved, we could write 1 for a win, 0 for a draw and −1 for a loss. We now repeat this process for every player in the game.

If we are dealing with a game of skill for which a mathematical theory exists, we will now be able to lay down a 'strategy' for a player. This is a list of instructions for a particular player telling him what to do in all conceivable circumstances which may arise in the game. The set of all possible lists of instructions for a player is called his 'strategy set.'

Games of skill fall into two classes: those such as chess or draughts in which the outcome is automatically determined once each player has chosen his strategy and those like poker or bridge where chance alone decides the outcome *once the strategies have been chosen.* In the first case, the value of the outcome can be immediately calculated once the strategies have been chosen; in the second case, elementary probability theory enables us to calculate the average or expected value of the game after the strategies are revealed.

To illustrate these ideas we first look at a game of chance in which two players, A and B, each toss a coin.

A\B	H	T
H	3	−2
T	−2	1

Figure 1

If the coins match and show heads, A receives 3c from B; if they match and show tails, A receives 1c from B; while if they do not match, A pays B 2c. The payoff to A for each possible outcome is tabulated in Figure 1. Played in this way the game is simply a game of pure chance, the players being unable to affect the outcome in any way. The average payoff to A will be

$$
\begin{aligned}
& 3 \,.\, \text{Prob (A \& B both get H)} \;+\; 1 \,.\, \text{Prob (A \& B both get T)} \\
& - 2 \,.\, \text{Prob (A gets H \& B gets T)} \;-\; 2 \,.\, \text{Prob (A gets T \& B gets H)} \\
& = 3 \,.\, \tfrac{1}{4} + 1 \,.\, \tfrac{1}{4} - 2 \,.\, \tfrac{1}{4} - 2 \,.\, \tfrac{1}{4} \\
& = 0.
\end{aligned}
$$

The game is therefore a fair one.

Now suppose that instead of tossing the coins, the player can choose which face of his coin to show. In this case A might well think "I can get a maximum of 3c by showing H if B shows H also. But B will guess that I'll think this and thus play T and then I will lose 2c. So if I'm smart I'll play T instead and get 1c. However B might be even more cunning and allow for this by playing H, so I will still lose 2c. So if I think that B will think that I'll think that he'll think that I'll play H, then I should play H anyway. But B might guess that I'll think that he'll think . . ." In order to avoid this never ending chain of reasoning A must abandon any idea of discriminating between his choices of heads or tails and use some random method of selection. Suppose A plays heads with probability p (and therefore plays tails with probability 1−p) and B plays heads with probability q. Then the average payoff to A will be

$$
\begin{aligned}
P(p,q) &= 3pq + 1(1\text{-}p)(1\text{-}q) - 2p\,(1-q) - 2(1-p)q \\
&= 8(3/8-p)(3/8-q) - 1/8
\end{aligned}
$$

If B chooses q = 3/8, he will ensure that he wins an average of at least 1/8c per game. On the other hand if A chooses p= 3/8, he will ensure that he loses no more than 1/8c per game on the average. Consequently B's best course of action is to choose heads with probability 3/8 and tails with probability 5/8. He can arrange this by having 3 objects labelled H and 5 labelled T and picking one at random before he makes his choice. A's best course of action in this case is exactly the same as that of B. Played in this way the game is advantageous to B, since he wins at least 1/8c per game in the long run.

The second illustration is thus a game of skill in which chance decides the outcome, once A and B have picked their strategies. For an elementary example of a game of skill in which the choices of strategies completely determines the outcome, take the game of "Scissors cut paper, paper wraps stone, stone blunts scissors."

Unfortunately, in practice, knowledge of the appropriate theory may not give you a sure method of maximising your gains. In chess and draughts, for instance, only the simplest strategies can be described in a reasonable time, while the number of possible strategies is so large that it is impossible to count them.

The usefulness of the mathematical theory of games is not restricted to recreational games, but extends to social conflicts and disputes as well. Industrial, political and military conflicts in which the goals of the contestants can be given numerical values, may all be regarded as games to which the theory applies. Typical of the military applications is the contest between a bomber and pursuing enemy missiles. The bomber pilot wants to know how best to manoeuvre his plane in order to avoid being hit. The guider of the missiles on the other hand, wants to guide them so as to maximise the probability of a hit. Although this problem is very complicated, present-day computers with the aid of the theory of games, are almost capable of providing answers to it (whether we should ask them to, is another matter).

Further Reading

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J.D. Williams, *The Compleat Strategyst.* McGraw-Hill Book Co., New York (1954).

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