PARTITIONS

In how many different ways can 4 be expressed as a sum of positive integers? The answer depends on whether we count sums like 1 + 3 and 3 + 1 as distinct or not; both kinds of problems are interesting. If we count them as distinct then we have the following eight possibilities:

$$1+1+1+1, \quad 1+1+2, \quad 1+2+1, \quad 1+3, \quad 2+1+1, \quad 2+2, \quad 3+1, \quad 4.$$

On the other hand if we regard the order in which we write the summands as being irrelevant, we only have five distinct possibilities:

$$1+1+1, \quad 1+1+2, \quad 1+3, \quad 2+2, \quad 4.$$

(Observe that 4 itself is regarded as a "sum," with only one summand.)

Generally, given a positive integer n, we want to know in how many different ways can it be expressed as a sum of positive integers. In other words, we want the number of distinct solutions of the equation

$$n = x_1 + x_2 + \ldots + x_r \qquad (1)$$

where x_1, x_2, etc. are positive integers and the number of summands, r, is arbitrary (possibly 1). Curiously, the first of the two problems, namely when two solutions which differ only in the order of terms are counted as distinct, is the easier one. Indeed if F(n) denotes the number of these solutions, then the number of those solutions in which $x_1 = 1$ is F(n–1), those in which $x_1 = 2$ is F(n–2), generally those in which $x_1 = k$ for a fixed positive integer $k < n$ is F(n–k), since then $n = k + x_2 + \ldots + x_r$, $n-k = x_2 + \ldots x_r$ and the number of these solutions is F(n–k). Now x_1 must necessarily take one of the values 1, 2, . . ., n, and these being mutually exclusive we have the formula

$$F(n) = F(n-1) + F(n-2) + \ldots + F(1) + 1, \quad n \geqslant 1, \qquad (2)$$

the last term 1 coming from the single solution $r = 1$, $x_1 = n$.

From this formula we can easily determine F(n); for, writing n + 1 instead of n we obtain

$$F(n+1) = F(n) + F(n-1) + \ldots + F(1) + 1$$

and subtracting from it the previous equation we get

$$F(n+1) - F(n) = F(n), \quad F(n+1) = 2F(n).$$

Now F(1) = 1 therefore F(2) = 2, F(3) = 2F(2) = 2^2 etc., generally $f(n) = 2^{n-1}$ (by induction, if you like). This checks with F(4) = 8 that we found earlier.

The second problem when the order of summands is irrelevant is more interesting but much more difficult. We may assume that the summands are arranged in an order of increasing magnitude and we are looking for solutions of equation (1) with the side condition

$$1 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_r.$$

We call such a solution a *partition* of n and denote the number of distinct partitions of n by P(n). By enumerating all possibilities one finds easily P(1) = 1, P(2) = 2, P(3) = 3, P(4) = 5 (as we have seen earlier), P(5) = 7 (namely 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 2 = 1 + 1 + 3 = 1 + 2 + 2 = 1 + 4 = 2 + 3 = 5), P(6) = 11 etc., but this sequence does not suggest any simple rule for the general term of the sequence. There is a comparatively simple recursive formula which can be used for the calculation of P(n), namely

$$P(n) = P(n-1) + P(n-2) - P(n-5) - P(n-7) + \ldots \qquad (3)$$

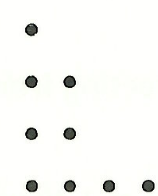
where the general form of the expression on the right is

$$(-1)^{k-1}\,[P(n - \tfrac{1}{2}k(3k-1)) + P(n - \tfrac{1}{2}k(3k+1))]\,, \quad k = 1, 2, 3, \ldots,$$

but the proof of this formula is much more difficult than the proof of (2) and there seems to be no simple expression for P(n) which will satisfy equation (3). There is an exceedingly complicated expression due to the famous British mathematician G.H. Hardy and the equally famous Indian mathematician S. Ramanujan but even to write down the formula here with explanations would be practically impossible and we shall not attempt it.

There are many other interesting problems concerning partitions, for instance we might be interested in the number of partitions in odd parts such as (in the case of n = 4) 1 + 1 + 1 + 1, 1 + 3; or unequal parts such as 1 + 3, 4; or m parts such as 1 + 3, 2 + 2 when m = 2; or in parts with the greatest part equal to m such as 1 + 1 + 2, 2 + 2 when m = 2, etc. Looking at the partitions of 4 or 5 we notice that the number of partitions into m parts is always equal to the number of partitions with the largest part equal to m. This can be proved generally as follows:

Represent partitions by rows of dots as shown in the figure which represents the partition 1 + 2 + 2 + 4 of 9 (the total number of dots is equal to n and the number of dots in each row is equal to the parts in which n is partitioned). Now to each partition of n there corresponds a "conjugate" partition, obtained by reading (from right to left) the number of dots in the vertical columns. For



instance the conjugate of the previous partition is 1 + 1 + 3 + 4. The length m of the bottom row gives the size of the largest part, and this is clearly equal to the number of columns in the diagram, that is the number of parts in the conjugate partition. Thus we have established a one to one correspondence between partitions of n with largest part equal to m, and partitions of n into exactly m parts. This proves the theorem.

The following beautiful result is also true: The number of partitions into unequal parts is equal to the number of partitions into odd parts. For instance 5 has three partitions into odd parts: 1 + 1 + 1 + 1 + 1, 1 + 1 + 3, 5, and three partitions into unequal parts: 1 + 4, 2 + 3, 5. We shall again describe a one to one correspondence between the two kinds of partitions.

Let $n = x_1 + x_2 + \ldots, 0 < x_1 < x_2 < \ldots$ be a partition into unequal parts. Now every x_i can be written uniquely as $2^{s_i} m_i$ where m_i is odd and $s_i \geqslant 0$, so we have

$$n = 2^{s_1} m_1 + 2^{s_2} m_2 + \ldots$$

with no two terms equal. For instance $2 + 5 + 8 = 2^1.1 + 2^0.5 + 2^3.1$ is a partition of n = 15 into unequal parts. Collect all the terms with $m_i = 1$; their sum will be either 0 (if no m_i is 1) or of the form

$$(2^{a_1} + 2^{b_1} + 2^{c_1} + \ldots).1, \quad 0 \leqslant a_1 < b_1 < c_1 < \ldots.$$

But the expression in brackets is just the binary representation of some positive integer k_1 and it can be written as $k_1.1$. Similarly, collecting the terms with $m_i = 3$ we get either 0 or $(2^{a_2} + 2^{b_2} + \ldots).3 = k_2.3$ with $k_2 > 0$, etc. Hence corresponding to the partition in unequal parts we obtain a solution of the equation

$$n = k_1.1 + k_2.3 + k_3.5 + \ldots \quad \text{with } k_1 \geqslant 0.$$

But this can be written as

$$n = \underbrace{1 + 1 + \ldots + 1}_{k_1 \text{ times}} + \underbrace{3 + \ldots + 3}_{k_2 \text{ times}} + \underbrace{5 + \ldots + 5}_{k_3 \text{ times}} + \ldots$$

which is a partition into odd parts. For instance in the previous example we obtain

$$15 = (2^1 + 2^3).1 + 0.3 + 2^0.5 = 10.1 + 0.3 + 1.5$$
$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 5.$$

A little reflection will show that this procedure establishes a one to one correspondence between the two kinds of partitions. For instance in the case of n = 6, the partition 1 + 2 + 3 = (1 + 2).1 + 1.3 corresponds to 1 + 1 + 1 + 3, 2 + 4 =

(2 + 4).1 corresponds to 1 + 1 + 1 + 1 + 1 + 1, 1 + 5 = 1.1 + 1.5 corresponds to itself (it is a partition both into unequal and into odd parts), and finally 6 = 2.3 corresponds to 3 + 3.

For further reading the advanced reader may consult 'An Introduction to the Theory of Numbers' by G.H. Hardy and E.M. Wright, chapter 19.

Problems

1. Write down the 11 distinct partitions of 6.
2. Prove that the number of partitions of n into m distinct parts is equal to the number of partitions of n—m into m parts.
3. Determine the number of partitions of n
   (a) into two parts,
   (b) into three parts.

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Activity: The Platonic Solids

In the class of solids known as polyhedra there are five which stand out for their complete regularity. These are known as the Platonic solids. As their name suggests they have been known to man for thousands of years, and it is not such a difficult job to prove that only five of them do exist. (See Parabola Vol 7 No 3).

You will notice in the diagrams and net diagrams on following pages the regular features of these solids, they are:—

1. Each face has the same number of edges.
2. Each vertex has the same number of edges joined to it.
3. All edges are of equal length.

PLATONIC SOLID	No. of edges per face	No. of edges per vertex
Tetrahedron	3	3
Hexahedron (cube)	4	3
Octahedron	3	4
Dodecahedron	5	3
Icosahedron	3	5

An interesting activity for you is to copy the net diagrams from following pages onto a piece of hard cardboard (empty 'Corn flakes' packets are quite good) and then cut out around the external lines. The internal lines are fold lines. You need to be very accurate in both drawing and cutting out the nets, to end up with a good set of platonic solids. If you enjoy this sort of activity, please let us know as there many other solids with interesting properties, that we can construct.