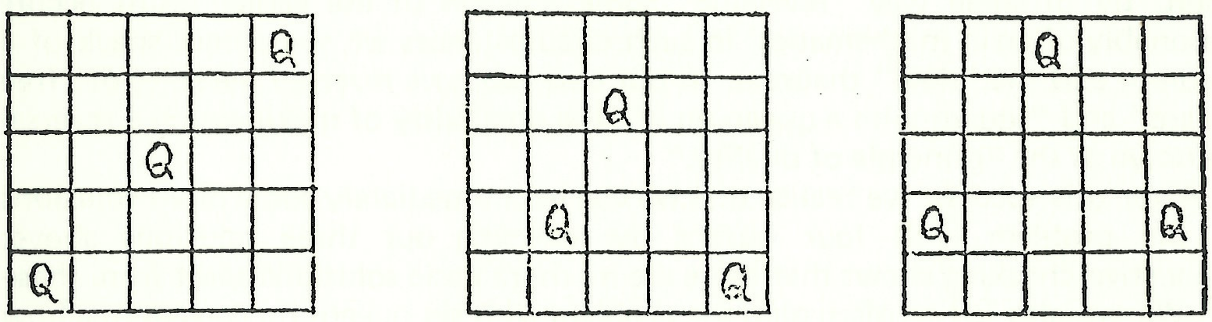
A FIVE-BY-FIVE CHESSBOARD

Two of the interesting problems related to chess are the number of queens needed to dominate a chess board (i.e. can be placed on the board in order to attack every square) and how few squares can be attacked by a given number of queens. As these problems are quite difficult with a regular chessboard, let us consider smaller boards instead.

It is easy to see that if we used a board three squares by three, one queen can either dominate the whole nine squares (from the centre) or cover all but two of the squares. The case of the four-by-four chessboard is left for you to work out yourself (see the problems at the end). The five-by-five board can be dominated completely by three queens (and so obviously by any greater number of queens) – Figure 1 gives three examples.

Figure 1



We now look to see how small we can make the territory that the three queens can cover. By trial and error, we find that we can place them so that as many as five squares are not attacked, as is shown in Figure 2; we also find that this is the only such formation (except for equivalent ones obtained by turning it around or turning it over). With four queens we find we can leave up to four squares free. Some of the possible ways of doing this are shown in Figure 3.

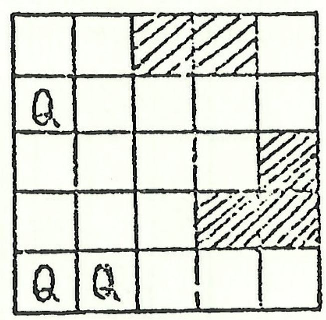


Figure 2

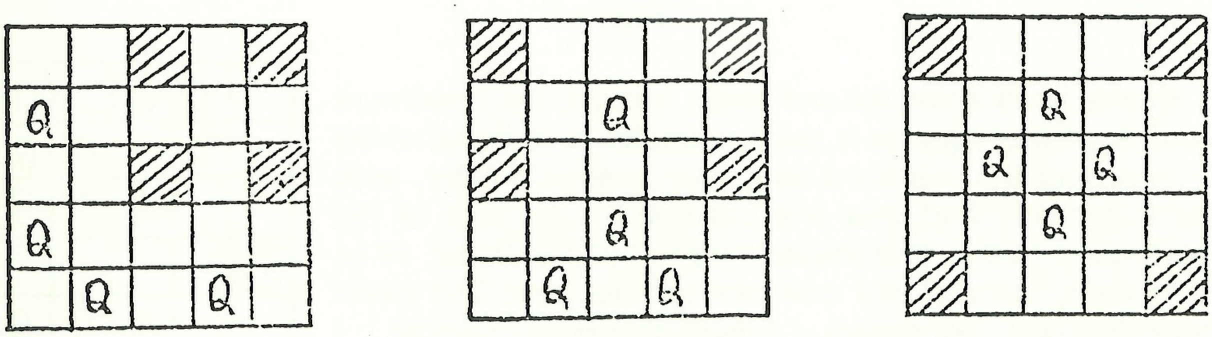


Figure 3

The problem seems to become more difficult when we use five queens as the number of possible formations is so large. It is not so hard to get positions which leave one or two squares free, but it is not easy to do better or to prove that we cannot do better. A small flash of insight reveals that we have essentially solved the problem already. Look back to the solution for the three queens and consider one of the free squares. It is not under attack by any of the three queens; hence, if we put a queen on this free square, it would not attack any of the three squares on which the queens are placed. As this argument holds for all of the five free squares, queens placed on them would not attack those three squares. Thus we have arrived at the only solution of the problem for five queens (Figure 4).

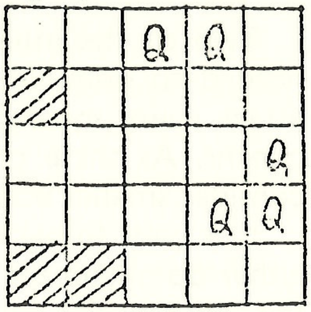


Figure 4

This process of getting a result which is new, but which is similar to an earlier result, by in some way "reversing" some features of the earlier result occurs reasonably often in mathematics; in such circumstances we sometimes speak of a theorem and the "dual" theorem. In our case, we have reversed the roles of "free square" and "square with a queen on it"; the possibility of making such a reversal is known as the "principle of duality."

After this success, we realise that we can get immediately some more solutions to the problem with four queens by reversing our three solutions above; exhaustive checking shows that there are no more basic solutions apart from these six. We are also led to discussing the problem with six queens by considering how many free squares we can get with two queens. By trial, the maximum is seven, and this occurs only in the four basic cases shown in Figure 5.

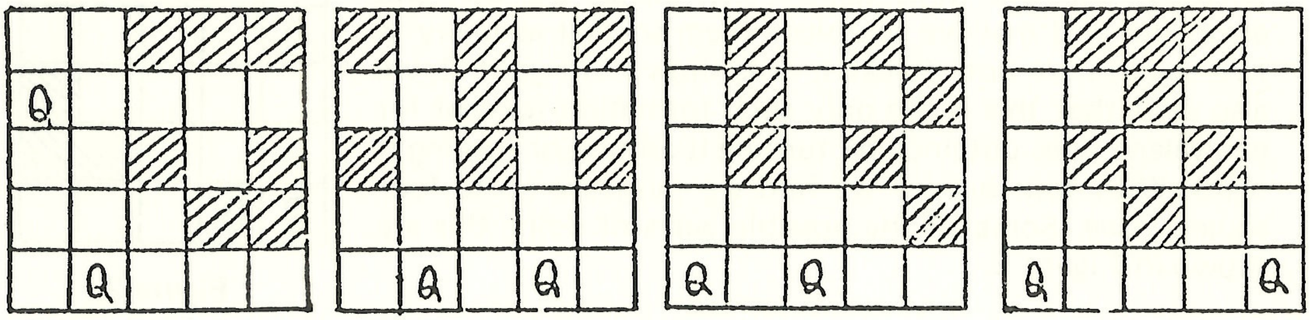


Figure 5

Hence, with both six and seven queens, the maximum number of free squares is two; examples may be obtained by reversing the solutions with two queens. Finally, with one queen, we find that any position on the edge of the board gives twelve free squares and so, with 8, 9, 10, 11 or 12 queens, we can still contrive to have one, but never more than one, free square — Figure 6 gives an example.

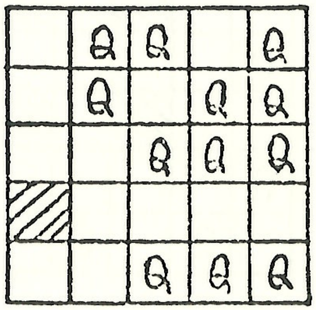


Figure 6

Problems

1. Prove that two queens cannot dominate our five-by-five chessboard.
2. Prove that two queens are needed to dominate a four-by-four chessboard.
3. Find the two different ways in which we can place two queens on a four-by-four board so as to leave three squares free.
4. What is the largest number of free squares we can get if we put three queens on a four-by-four board?
5. What is the largest number of queens we can put on a four-by-four board so as to leave one free square?
6. Try the equivalent problems for a six-by-six board, and then for a seven-by-seven board, and then . . .
7. Perhaps you would also like to try the rather easier problems where you use rooks, or bishops, in place of the queens?

NOTE: Although a computer could be programmed to give all the solutions to all these problems, it could not discover the duality we have found: for that, a human brain is needed.

Doug Mackenzie

Mr Mackenzie is a member of the lecturing staff at the University of NSW. Some of our readers might like to make comments on the Note at the end of his article — Editor.

☆ ☆ ☆

Tetrahedron {3, 3}