GEOMETRY – ALL DONE WITH MIRRORS!

When we do Euclidean geometry, we use the idea of congruence of triangles and we have certain rules ("three sides," "2 sides and included angle," or "2 angles and a side") for deciding whether two triangles are congruent or not. But what do we really mean by the idea of congruence of triangles?

One way of thinking of congruence is "being able to pick one triangle up and place it on top of the other so that it fits everywhere." In other words, we should be able to transform one triangle into the other without altering the lengths of any segments (e.g. by stretching). Such a transformation is called a *congruence transformation* and examples are:

(a) sliding all figures a certain distance in some direction (called a *translation* of the plane along a vector),

(b) rotating the plane through some angle around a certain point (not necessarily the origin!),

(c) reflecting the plane in a mirror along a certain line,

(d) doing nothing (after all no lengths are changed!) – called the identity transformation.

Some of these are illustrated in the article *Polyominoes* on page 2 and in Figure 1 below.

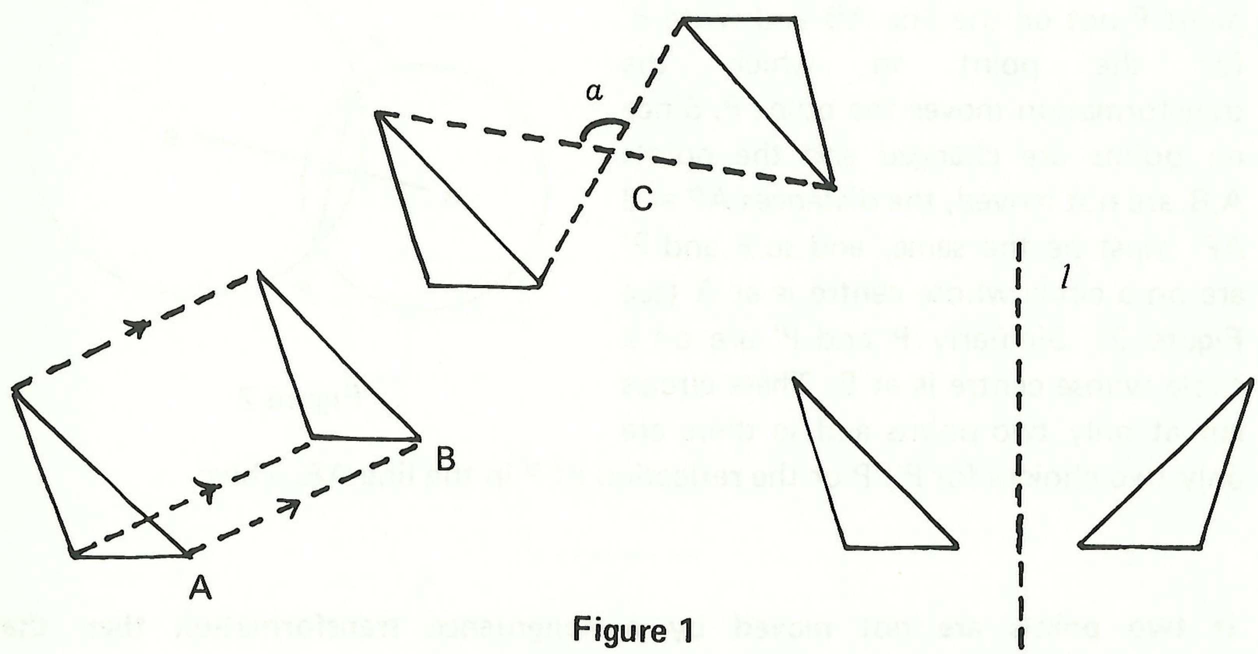

Figure 1

Are these examples the only kinds of congruence transformation? One obvious way of inventing a new transformation would be to use two of the examples together, e.g. first translate the plane and then rotate it about some given point, or rotate it about two given points one after the other. If we use the symbol D_{AB} for the translation along the vector $\overrightarrow{\mathrm{AB}}$, $T_{\alpha,\mathrm{C}}$ for the rotation through an angle α about the point C and S_l for the reflection in the line l, we can use this method to get transformations with the symbols

$$D_{\mathrm{AB}}D_{\mathrm{CD}},\ D_{\mathrm{AB}}T_{\alpha,\mathrm{C}},\ D_{\mathrm{AB}}S_l,\ T_{\alpha,\mathrm{C}}D_{\mathrm{AB}},T_{\alpha,\mathrm{C}}T_{\beta,\mathrm{D}},\ T_{\alpha,\mathrm{C}}S_l,\ S_lD_{\mathrm{AB}},\ S_lT_{\alpha,\mathrm{C}},S_lS_m.$$

In the last case, by using two mirrors, it is easy to see that S_lS_l is the identity transformation and S_lS_m is a translation when $l \parallel m$ or a rotation around C if l and m meet at C; also in the first case you can use vector addition to find $D_{\mathrm{AB}}D_{\mathrm{CD}}$. Some of the others are a little harder but you might like to try them.

Let us attack the question of what congruence transformations are like from another point of view. We know that if none of the points of the plane are moved, the transformation is the identity. What if we only know two points A,B are not moved? For convenience, let us choose a point P not on the line AB and write P′ for the point to which the transformation moves the point P. Since no points are changed and the points A,B are not moved, the distances AP and AP′ must be the same, and so P and P′ are on a circle whose centre is at A (see Figure 2). Similarly P and P′ are on a circle whose centre is at B. These circles cut at only two points and so there are only two choices for P′: P or the reflection of P in the line AB. Thus:

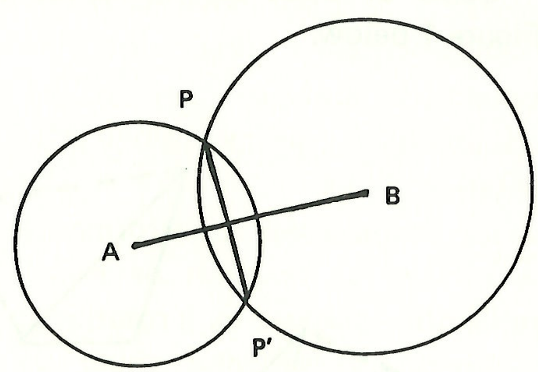

Figure 2

If two points are not moved by a congruence transformation then that transformation is either the identity or a reflection.

Now let us suppose that there is one (only) point A not moved. If B is another point, let us write B′ for the point to which B is moved. Then B′ is not B or A. (Why?) Let l be the line which bisects the angle $\angle$BAB′ (see Figure 3). Then the transformation can be achieved by first reflecting in l and then using another congruence transformation A. Since both S_l and the original transformation S_lA move B to B′ and do not move A, the transformation A must not move A or B′ and so is a reflection. Thus:

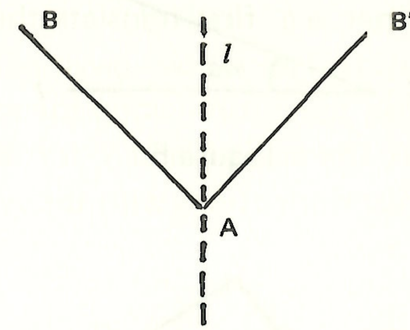

Figure 3

If one and only one point is not moved by a congruence transformation, then that transformation consists of two reflections in intersecting lines (and so is a rotation).

We are nearly there – now we only have to look at a congruence transformation which moves every point to a new point. Suppose it moves a point A to another point A′ and let l be the perpendicular bisector of the segment AA′. Thus S_l also moves A to A′ and again the given transformation can be written S_lA where A does not move A′. If there is another point which A does not move, it is a reflection and so S_lA is a translation. Otherwise A only moves one point and so is a rotation. Thus:

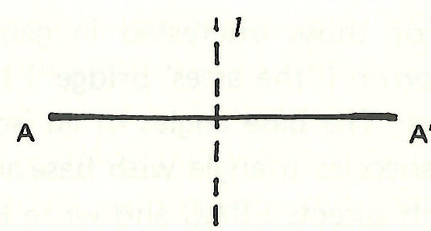

Figure 4

Every congruence transformation is (i) a reflection or (ii) consists of two reflections (a rotation or translation) or (iii) consists of three reflections (a reflection followed by a rotation).

There is still one thing to clear up: what does the last congruence transformation $S_lT_{a,C}$ look like? If the point C is on the line l, draw a line m through C at an angle of ½a to l (see Figure 5). It is easy to show (using mirrors) that S_lS_m is $T_{a,C}$. Thus $S_lT_{a,C} = S_lS_lS_m = S_m$ (since S_lS_l is the identity) and so $S_lT_{a,C}$ is a reflection. If C is not on the line l, draw the line $m \parallel l$ through the point C (see Figure 6). Using the fact that S_mS_m is the identity, we can see that $S_lT_{a,C} = S_lS_mS_mT_{a,C}$ where S_lS_m is a translation and $S_mT_{a,C}$ is a reflection. This kind of transformation is called a glide reflection and is illustrated in Figure 7 over page:

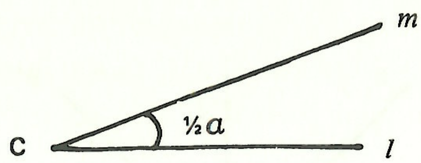

Figure 5

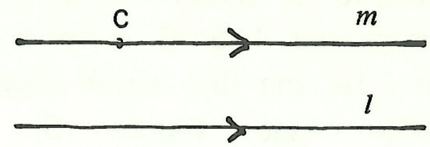

Figure 6

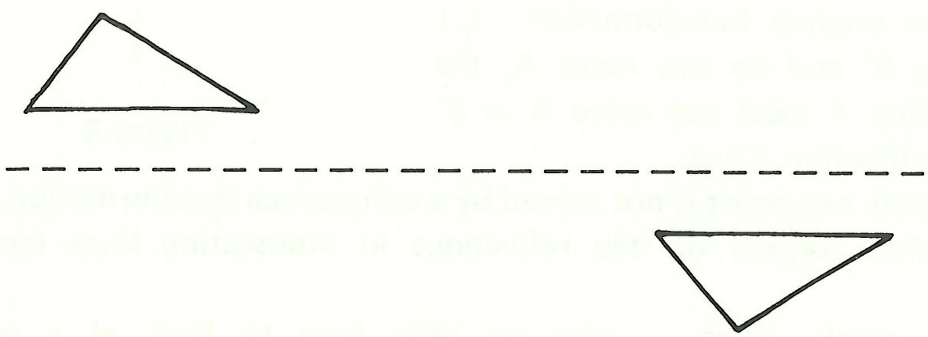

Figure 7

For those interested in geometric theorems there is a theorem called *pons asinorum* ("the asses' bridge") for which Euclid gives a very complicated proof. It states: **The base angles of an isosceles triangle are equal.** To prove it, let ABC be an isosceles triangle with base angles B and C. Reflect the triangle in the line AD which bisects ∠BAC and write B′, C′ for the points to which B,C are moved (see Figure 8).

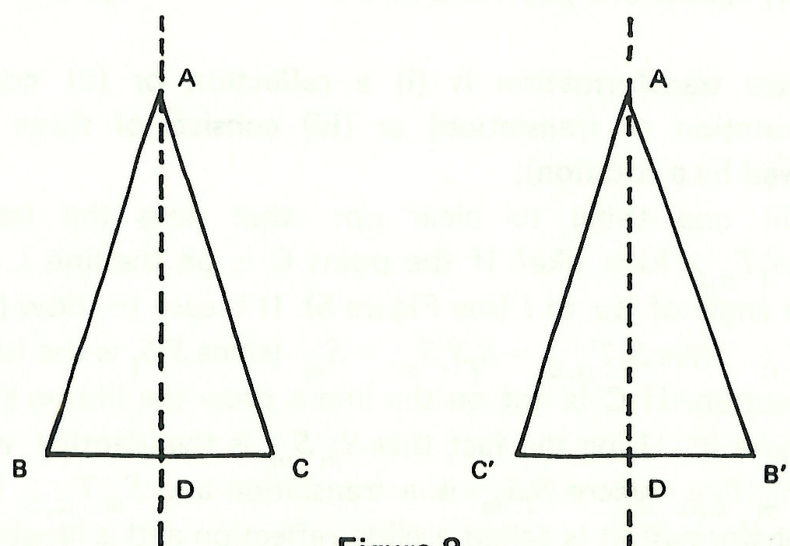

Figure 8

Since this is a congruence relation, $\Delta ABC \equiv \Delta A'B'C'$, and so $\angle B = \angle B'$, $\angle C = \angle C'$. But it is very easy to show that B′ is the point C and C′ is the point B, proving the theorem.

Anyone interested in reading further is encouraged to read "Introduction to Geometry" by H.S.M. Coxeter – a must for your school library!

Problems

1. Using vectors, show that $D_{AB}D_{BC} = D_{AC}$. Use this to find $D_{AB}D_{CD}$ in general.
2. Show that $T_{\alpha,C}T_{\beta,C} = T_{\alpha+\beta,C}$. What about $T_{\alpha,C}T_{\beta,D}$ where C,D are different points, where $\alpha + \beta = 180°$?
3. Every congruence transformation is a reflection, rotation, translation or glide-reflection. Identify all the products $D_{AB}D_{CD}$ etc. (including those with glide-reflections).
4. If P is the point (p,q) and is moved to the point P′ by a congruence transformation find the co-ordinates of P′ where the transformation is
   (a) a translation along the vector from (0,0) to (h,k).
   (b) a rotation through α about the point (a,b).
   (c) a reflection in the line $y = mx + c$.
5. *(For 1st level students)* Show that the set of all congruence transformations is a group. Have we used any group properties in this article?
6. C is a point inside the square ABDE and $\angle CDE = \angle CED$. What can you say about the triangle ABC?
7. Prove that any triangle having two equal medians is isosceles.

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GAME : Points & Paths

Here is an interesting game you can play with a friend.

Place three points on a piece of paper. To go, join any two points with a path and place a point somewhere on that path, with the proviso that no point shall have more than three paths joined to it. The last person to be able to go, wins. It is interesting to note that the game can last at most eight moves (why?), and so should be easily analysed. We look forward to your comments.