SOLUTIONS TO SCHOOL MATHEMATICS COMPETITION 1973

Junior Division

1. Given a triangle with sides a,b,c and a segment of length d. Denote by p the smaller of a and d, by q the smaller of b and d, and by r the smaller of c and d. Can you always construct a triangle with sides p,q,r?
   Give reasons for your answer.

Answer. A triangle can be constructed if, and only if, (i) $p + q > r$, (ii) $q + r > p$ and (iii) $r + p > q$. There are essentially 4 different cases.

(a) $p = q = r = d$. Then (i), (ii), (iii) are true.

(b) $p = q = d$, $r = c$. Then $p + q > d > c$, so (i) is true, while (ii), (iii) are clearly true.

(c) $p = d$, $q = b$, $r = c$. $p + q = d + b > c + b > c$ so (i) is true. $q + r = b + c > a > d$, so (ii) is true. $r + p = c + d > c + b > b$ so (iii) is true.

(d) $p = a$, $q = b$, $r = c$. We are given (i), (ii), (iii) true by the question.

Conclusion: A triangle can always be constructed.

Note that (b) and (c) each imply $a < b < c$. If we had, for example, $c < b < a$ then (b) would start: $r = q = d$, $p = a$ and (c) would start $r = d$, $q = b$, $p = a$, but the arguments would be exactly the same after we had interchanged p and r and c and a. Similarly for $b < a < c$ etc., while isosceles triangles give rise to no new case. For equilateral triangles we can have only (a) or (d).

2. Solve the system of simultaneous equations

$$x^2 + yz = 5$$
$$y^2 + xz = 3$$
$$z^2 + xy = 3.$$

Answer. $y^2 - z^2 + x(z - y) = 0$ or $(y - z)(y + z - x) = 0$.

Either (i) $y = z$ or (ii) $x = y + z$.

(i) gives $x^2 + y^2 = 5$, $y^2 + xy = 3$,

so $3(x^2 + y^2) - 5(y^2 + xy) = 0$

or $3x^2 - 5xy - 2y^2 = 0$
or $(3x + y)(x - 2y) = 0$.
Either (a) $x = 2y$ or (b) $y = -3x$.
(a) gives $5y^2 = 5$, $y = \pm 1$, $x = \pm 2$, $z = \pm 1$. That is we have either $x = 2$, $y = z = 1$, or $x = -2$, $y = z = -1$.
(b) gives $10x^2 = 5$, $x = \pm 1/\sqrt{2}$ so we have either $x = 1/\sqrt{2}$, $y = z = -3/\sqrt{2}$ or $x = -1/\sqrt{2}$, $y = z = 3/\sqrt{2}$.
(ii) gives $y^2 + 3yz + z^2 = 5$, $y^2 + yz + z^2 = 3$
so $3(y^2 + 3yz + z^2) - 5(y^2 + yz + z^2) = 0$
or $-2y^2 + 4yz - 2z^2 = 0$
or $y^2 - 2yz + z^2 = 0$.
So $y = z$ and $x = 2y$ giving $3y^2 = 3$ and the same 2 sets of solutions as in (i) (a).

3. (i) Given a regular hexagon, ABCDEF, how many triangles can be formed such that each of their vertices is also a vertex of the hexagon?

(ii) Of the triangles in part (i) how many will have their centroid lying on the diagonal AD?

Give reasons for your answers.

Answer (i). We need to know the number of different triplets of 3 letters, without regard to order, which can be chosen from 6 different letters. For those who know the formula concerned this is:

$$^6C_3 = \binom{6}{3} = \frac{6!}{3!3!} = \frac{6.5.4}{1.2.3} = 20$$

Otherwise we count first all the sets of 3 with A in them. This is equal to the number of different pairs that can be chosen from B, C, D, E, F. There are 4 pairs containing B, 3 containing C but not B, 2 containing D but not B or C and 1 containing E but none of B, C or D, making 10 in all.

Then use a similar method to count the number of triplets that can be formed from B, C, D, E, F; this is $3 + 2 + 1 = 6$. The number of triangles with vertices from C, D, E, F is $2 + 1$ and the number from D, E, F is 1.

So total $= 10 + 6 + 3 + 1 = 20$.

Answer (ii). Each main diagonal AD, BE, CF is an axis of symmetry so their point of intersection, O, is the centre of symmetry of the hexagon and OA, OB, OC, OD, OE, OF are equal. Also AD bisects BF, CE by symmetry. So, as the centroid of a triangle lies on each median, and OA (or part of it) is a median for each of the triangles ABE, ACF, ABF, ACE, these are 4 of the triangles sought. Similarly we get triangles DBE, DCF, DBF, DCE.

As AD must be a median for any triangle with A as vertex and centroid on AD, these 4 triangles mentioned are the only permissible ones with A as a vertex. Similarly for D. Of the triangles that can be formed from B, C, E or F only, take BCE as representative. (They must all be congruent – why?) CO is a median and O can't be the centroid. So BCE won't do and the only suitable triangles are the 8 given. ✫

4. An "arithmetic progression" is a sequence of numbers in which the difference between successive numbers is always the same. We call the number of terms in the arithmetic progression the length of the progression. For example, 1, 25, 49 is an arithmetic progression with constant difference 24 and of length 3 while 5, 8, 11, 14, 17, 20, . . . , $(5 + 3n)$, . . . is an infinite arithmetic progression with constant difference 3.

   (i) Show that it is impossible to extract from the set of all perfect squares
$$\{1, 4, 9, 16, 25, 36, 49, \ldots\}$$
   a sequence of numbers which forms an infinite arithmetic progression.

   (ii) Show how to partition the set of all positive integers into two disjoint sets A and B neither of which contains an infinite arithmetic progression. (Sets A and B are called disjoint if they have no element in common.)

Answer (i). The essential point here is that the squares get increasingly further and further apart as their magnitude increases. Consider n and n + 1, two consecutive integers. Then $(n + 1)^2 - n^2 = 2n + 1$. $[= (n) + (n + 1)]$.

If x^2 is the first term of the arithmetic progression and d the difference between successive terms, $d < (d + 1)^2 - d^2$, so that when $y \geqslant d$, we cannot possibly have $z^2 - y^2 = d$. No matter how big d may be we must eventually reach this position in any such contemplated progression.

Answer (ii). One method is as follows:–

Let n be any integer greater than 1. Put into A the following sets of consecutive integers: $\{1, 2, \ldots, n-1\}$, $\{n^2, n^2+1, \ldots, n^3-1\}$, $\{n^4, \ldots, n^5-1\}$ and, in general, the set $\{n^{2k}, n^{2k+1}, \ldots, n^{2k+1}-1\}$. The remainder of the positive integers form B. This works for much the same reason mentioned in (i) as the subsets get further and further apart, the bigger k becomes. ☆

5. At an inaugural meeting of a newly formed society the following fact is observed about any three members A, B and C. If A and B know each other and B and C also know each other, then C knows no person in the Society except B. Prove that the members of the Society can be placed in two rooms so that no two people in the same room know each other.

Answer. We note from the data that A must be in the same position as C; he also doesn't know anybody other than B.

We therefore start by putting into Room 1 those people who know more than one other member. These are like B so that the people whom they know do not know anybody else. Consequently none of the B's can know each other.

Next we put into Room 2 all the people known to those in Room 1. (A's or C's) Among the remaining members those who know exactly one other member clearly form separate pairs. All we need to do with them is to put one from each pair into Room 1 and the others into Room 2. The members remaining will be those who know nobody. These we can allocate however we like. ☆

Senior Division

1. Given $a > 0$, $a \neq 1$, solve the following equation for x:

$$\sqrt{\log_a \sqrt[4]{(ax)} + \log_x \sqrt[4]{(ax)}} + \sqrt{\log_a \sqrt[4]{(x/a)} + \log_x \sqrt[4]{(a/x)}} = a.$$

Answer. As $\sqrt{y}$, and hence $\sqrt[4]{y}$, is only defined for $y > 0$, we must have $ax > 0$ and hence $x > 0$.

Let $t = \log_a x$: then $\log_x a = 1/t$.

Now $\log_a \sqrt[4]{(ax)} + \log_x \sqrt[4]{(ax)}$

$$\begin{aligned}
&= \tfrac{1}{4}\log_a a + \tfrac{1}{4}\log_a x + \tfrac{1}{4}\log_x a + \tfrac{1}{4}\log_x x \\
&= \tfrac{1}{4} + \tfrac{1}{4}t + \tfrac{1}{4}.1/t + \tfrac{1}{4} \\
&= \tfrac{1}{4}(t + 2 + 1/t) \\
&= (t+1)^2/4t.
\end{aligned}$$

Similarly, $\log_a \sqrt[4]{(x/a)} + \log_x \sqrt[4]{(a/x)} = (t-1)^2/4t$.

Since both these expressions must be positive, we must have $t > 0$ and so $a = (|t+1| + |t-1|)/2\sqrt{t}$.
We either have
$a = (t+1+t-1)/2\sqrt{t}$ and so $t = a^2$ or
$a = (t+1+1-t)/2\sqrt{t}$ and so $t = 1/a^2$.
In the first case, $\log_a x = a^2$ and so $x = a^{a^2}$; in the second case, $\log_a x = 1/a^2$ and so $x = a^{1/a^2}$. ☆

2. a, b, c, d are positive integers such that $ab = cd$. Prove that $a + b + c + d$ and $a^2 + b^2 + c^2 + d^2$ are not prime numbers.

Answer. We may assume that a, b, c, d have no common factor. (Otherwise this factor will divide $a + b + c + d$ and $a^2 + b^2 + c^2 + d^2$).

Let e be the greatest common divisor of a and c (written (a,c)), and let f be (a,d), g be (b,c) and h be (b,d).

For any prime number p, suppose p^A is the largest power of p dividing a, p^B is the largest power of p dividing b, . . . , p^H is the largest power of p dividing h. But $A + B = C + D$ and at least one of A,B,C,D is zero. Without loss of generality assume $A = 0$. Then $E = F = 0$, $G = C$, $H = D$. So $A = E + F$, $B = G + H$, $C = E + G$ and $D = F + H$. Consequently $a = ef$, $b = gh$, $c = eg$ and $d = fh$.

Therefore $a + b + c + d = ef + gh + eg + fh$
$= (e + h)(f + g)$.

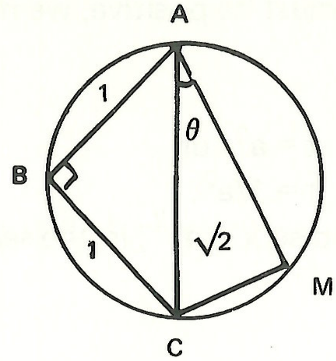
Now all of e, f, g, h are $\geqslant 1$
$e + h$, $f + g$ are $\geqslant 2$
$a + b + c + d$ is not a prime.

Similarly if n is a positive integer, $a^n + b^n + c^n + d^n$
$= e^n f^n + g^n h^n + e^n g^n + f^n h^n$
$= (e^n + h^n)(g^n + f^n)$
and $a^n + b^n + c^n + d^n$ is not a prime.

In particular if $n = 2$, $a^2 + b^2 + c^2 + d^2$ is not a prime. ☆

3. The triangle ABC is isosceles and has a right angle at B. The point M is on the circumcircle of ABC. Find the position of all points M such that is possible to construct a triangle from the segments BC, MA, MC.

Answer. A triangle is constructible from 3 line segments if the length each of the segments is less than the sum of the other 2.



Let length BC = 1 unit.
Clearly the centre of circumcircle is the midpoint of AC.
Let M be arbitrary on the circle. Let $\angle CAM = \theta$.

Now as M varies over the circle, θ varies from $-\pi/2$ to $+\pi/2$. Here we need only find values of θ for which the condition holds. Indeed since any position of M is by symmetry equivalent to 3 others we need only solve for $0 \leqslant \theta \leqslant \pi/4$.

Now if $\theta = \pi/4$, AM = CM = BC the triangle is equilateral and can be constructed. If θ is very close to 0

$$AM \cong \sqrt{2} \quad CM \cong 0 \quad BC = 1$$

and no triangle can be constructed.

Now if $\theta < \pi/4$, CM < BC < AM. Hence condition is simply

$$CM + BC \geqslant AM.$$

As θ varies from $\pi/4$ to 0, AM continuously increases in size and CM continuously decreases in size, hence we need only find θ such that

$$CM + BC = AM.$$

Now BC = 1, AC = $\sqrt{2}$, AM = $\sqrt{2}\cos\theta$, CM = $\sqrt{2}\sin\theta$ (since $\angle AMC = \pi/2$).

Solve $\sqrt{2}\sin\theta + 1 = \sqrt{2}\cos\theta$

$$-\sin\theta\,(\tfrac{1}{2}\sqrt{2}) + \cos\theta\,(\tfrac{1}{2}\sqrt{2}) = \tfrac{1}{2}$$

$$\sin(\pi/4 - \theta) = \tfrac{1}{2}$$

$$\pi/4 - \theta = \sin^{-1}\tfrac{1}{2} = \pi/6.$$

$$\theta = \pi/4 - \sin^{-1}\tfrac{1}{2} = \pi/4 - \pi/6 = \pi/12.$$

Conclusion: Triangle is constructible if $\pi/12 \leqslant \theta \leqslant 5\pi/12$ or $-5\pi/12 \leqslant \theta \leqslant -\pi/12$.

4. (i) Find a set A of 6 non-negative integers and an infinite set B of non-negative integers with the property that every non-negative integer can be written in exactly one way as a sum a+b where a is an element of

A, b is an element of B.

(ii) Find two such sets, each containing infinitely many elements.

Answer (i). Let A = {0,1,2,3,4,5}. Let B = {0,6,12,18,24, . . .}.
Then every integer can be written as its remainder on dividing by 6 plus a multiple of 6 in a unique way.

Generalization. Clearly if n is an arbitrary positive integer and A = {0,1,2,. . . ,n−1}, B = {0,n,2n,3n,. . .} then every integer is uniquely expressible as

$$a + b, \quad a \in A, \quad b \in B.$$

Answer (ii). Consider

$$869731 = 809030 + 60701.$$

In general write

$$n = n_0 + n_1.10 + n_2.10^2 + n_3.10^3 + \ldots + n_k 10^k$$

as

$$n = (n_0 + n_2.10^2 + n_4.10^4 + \ldots) + (n_1.10 + n_3.10^3 + n_5.10^5 + \ldots).$$

A = set of integers expressible in the form $a_0 + a_2.10^2 + a_4.10^4 + \ldots$

B = set of integers expressible in the form $a_1.10 + a_3.10^3 + a_5.10^5 + \ldots$

Generalization: There is nothing sacred about base 10! There is nothing sacred about dividing the positive integers into odds and evens! ☆

5. We have 30 locked boxes and 30 keys, each opening just one lock. Each box has an opening through which we throw one key at random.

   (i) We break one of the boxes. What is the probability that we can open all the other boxes without breaking them?

   (ii) If we initially break two boxes instead of one, what is the probability that we can open all the other boxes without breaking them?

   Give reasons for your answer.

Answer. With probability questions the best approach (often) is to ignore the nonsense about keys, boxes, etc. and attempt to reformulate the problem in a simpler form so that the answer becomes obvious!!!

(i) is equivalent to the following problem:

Put 1 red ball (= the key to the box initially broken open) and 29 blue balls into a bag. What is the probability that the red ball is the *last* one to be drawn out (drawing the balls out one at a time).

Solution: Probability = 1/30.

(ii). Same as (i) except there are now 2 reds and 28 blues.

The probability that the *last* drawn out is red = 2/30 = 1/15.

[*To the Reader:* Have you checked *in your own mind* that the reformulation is valid.]

Generalization: If n boxes are broken, probability is n/30.

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Problem Squares

One hundred digits are chosen at random and written down to form the two numbers m and n. What is the probability that $m = n^2$?

A Weather Report

In a certain place in NSW, the temperatures at 6 am for the past five mornings (to the nearest degree Celsius) have all been different. If the product of those temperatures is 12, what are they?

Squares and Cubes

Find the smallest two numbers such that the difference of their squares is a cube and the difference of their cubes is a square.

Answers on page 40.