question? What odds would you consider fair?

[This problem has already caused a great deal of debate in the Mathematics Department here – Ed.]

SOLUTIONS

Solutions to Problems 201–210 in Vol. 9 No. 1

The names of successful problem solvers appear after the solution to Problem 210.

Junior

J201 Find the last digit of 7^{1001}.

Answer The final digit of 7^4 is 1. Hence the same is true of 7^8 $7^{12}, \ldots 7^{1000}$. Hence $7^{1001} = 7 \times (7^{1000})$ must end with a 7.

J202 To divide 410,256 by 4, we need only take the 4 from the beginning and put it at the end; i.e. 410,256 ÷ 4 = 102,564. Find a number starting with 8 which can be divided by 8 by taking the 8 from the beginning to the end.

Answer

```
8 | 8 1 0 1 2 6 5 8 2 2 7 8 4
  ---------------------------
    1 0 1 2 6 5 8 2 2 7 8 4 8
```

The digits are obtained in the order indicated by the arrows, by simply carrying out the usual division process. As each extra digit in the quotient is obtained it is placed as the next digit in the dividend, and the division can be carried through one extra place. The process is halted when the digit 8 is obtained at the same time as a remainder of 0.

Intermediate

1203 If $a > b > 0$, which of $\sqrt{a-b}$ and $\sqrt{a} - \sqrt{b}$ is the greater?

Answer. Since the numbers are both > 0, the greater has the greater square. We have to decide which is greater of $a-b$ and $a-2\sqrt{(ab)}+b$.
Now $a > b > 0 \Rightarrow ab > b^2 \Rightarrow \sqrt{(ab)} > b \Rightarrow a-2b > a-2\sqrt{(ab)} \Rightarrow a-b > a-2\sqrt{(ab)}+b \Rightarrow \sqrt{(a-b)} > \sqrt{a} - \sqrt{b}$.

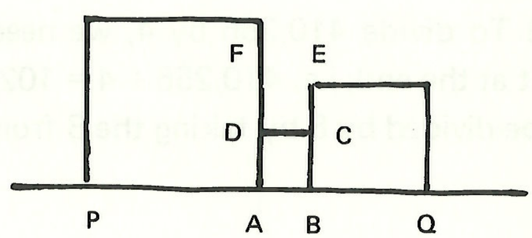
1204 (i) Prove that if it is possible to tile a rectangle with square tiles no two of which are of the same size then

(a) the smallest tile does not touch any side of the rectangle.

(b) the smallest tile touches exactly four other tiles.

Answer (i) (a) This is clear from the accompanying diagram where PQ is the edge of the rectangle. The larger squares on either side of the smallest square ABCD leave a space DCEF which obviously cannot be covered by tiles larger than ABCD. (The same difficulty arises if ABCD is placed in a corner of the rectangle to be tiled).

Figure 1



(b) The observation above applies equally if PQ is the edge of one or more tiles within the rectangle. It follows that the tile touching ABCD along AB must end at A or B. If it ends at B, the tile touching ABCD along BC must end at C and so on. The neighbouring tiles must therefore be as in Figure 2.

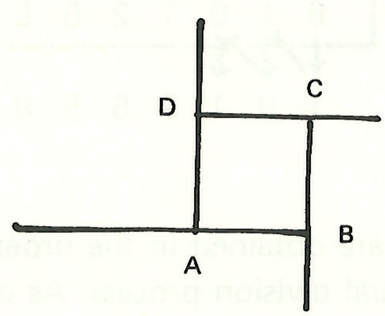


Figure 2

1204 (ii) This should have read:
Prove that it is NOT possible to exactly fill a rectangular box with cubes no two of which are of equal size.

Answer (ii) If it were possible, the cubes touching the bottom of the box would correspond to a tiling of that rectangle with unequal squares. The smallest of these cubes sits at the bottom of a "square well" formed by the surrounding larger cubes and so must have its upper face covered by smaller cubes. Similarly the smallest of these, in turn, sits at the bottom of a square well, requiring the existence of yet smaller cubes. Since the same argument may be repeated indefinitely, it is clear that the box can never be filled by any finite collection of different cubes.

I205 Suppose that the number ABCDE (where the letters denote decimal digits) is divisible by 271. Prove that also BCDEA is divisible by 271. (For example: since 27371 is divisible by 271, so is 73712).

Answer Let X be the number whose decimal representation is BCDE. We have to show that if 271 divides 10,000A + X it divides 10X + A.

If 271 divides 10,000A + X, we can write 271k for 10,000A + X.

Thus $2710k = 100{,}000A + 10X$

$= 99{,}999A + (10X + A)$

$= 271 \times 369A + (10X + A)$

So $10X + A = 271(10k - 369A)$ is divisible by 271.

Open

O206 Prove that if n is not divisible by 3 then $n^{13} - n$ is divisible by $2^{13} - 2$.

Answer Since $2^{13} - 2 = 2 \times 3^2 \times 5 \times 7 \times 13$ we are finished if we show that each of 2, 5, 7 and 13 divide $n^{13} - n$ for any n; and that 9 divides $n^{13} - n$ for any n not divisible by 3. We illustrate by indicating how to prove the last of these statements.

If n is not divisible by 3, it leaves, on division by 9, a remainder equal to one of 1, 2, 4, 5, 7 and 8. We use the fact that if x leaves remainder r_1, and y leaves remainder r_2 then xy leaves the same remainder as $r_1 r_2$. (Prove this).

It follows that n^2 leaves one of the remainders 1, 4, 7, 7, 4, 1; that n^3 leaves one of the remainders 1, 8, 1, 8, 1, 8; and finally that $n^6 = (n^3)^2$ leaves the remainder 1 for all the numbers n considered.

Hence n^{12} also leaves the remainder 1 on division by 9 whence n^{13} leaves the same remainder as n. That is, $n^{13} - n$ is divisible by 9, for all n not divisible by 3.

Only minor modifications of this method are required to prove, say, that $n^{13}-n$ is always divisible by 5. Thus, if $5|n$, the result is obvious. Otherwise n leaves the remainder 1 or 2 or 3 or 4 on division by 5, and calculation shows that n^4 must leave remainder 1. Hence so does n^{12}, and therefore n^{13} and n leave the same remainders. Thus $5|(n^{13}-n)$ for all n.

Comment. The solution is much shorter if we use Euler's Theorem:—

Let $\phi(n)$ be the number of positive integers less than n and having no factor in common with n. [For example, $\phi(9) = 6$ since 1, 2, 4, 5, 7, 8 are the only smaller positive integers relatively prime to 9.] Then, if a is relatively prime to n, $a^{\phi(n)}$ leaves the remainder 1 on division by n. When n = p, a prime number, all of 1, 2, . . . , p−1 are relatively prime to p, so $\phi(p) = p-1$ and the statement becomes a^{p-1} leaves the remainder 1 on division by p, for all integers a not divisible by p; which is known as Fermat's Theorem.

O207 ABCD is a quadrilateral such that

(i) $AC^2 + BD^2 = 2AB^2 + AD^2 + BC^2$

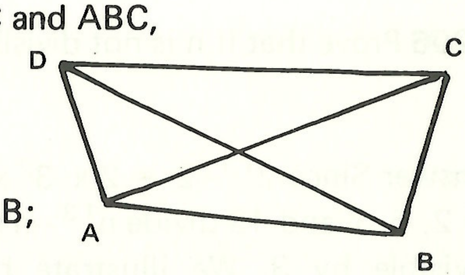
(ii) AD = BC.

Prove that it is a parallellogram.

Answer By the cosine formula, in triangles ABC and ABC,

$AC^2 = AB^2 + BC^2 + 2AB \,.\, BC \cos\angle ABC$

$BD^2 = AD^2 + AB^2 + 2AB \,.\, AD \cos\angle DAB.$



Adding, and using (i), gives

$0 = 2AB.BC \cos\angle ABC + 2AB.AD \cos\angle DAB;$

whence, using (ii), $\cos\angle ABC = -\cos\angle DAB$.

Further, both angles, being interior angles of triangles, are less than 180°. Hence, these angles are supplementary, which makes AD || BC.

Comment. At first glance the above looks like s satisfactory proof, but it isn't! To make it so the question should be more carefully posed. First, we should have made it clear that the problem was a *plane geometry* problem. (The final step obviously depends on this.)

There is however a more serious (since less obvious) omission in the statement, which renders the above "proof" invalid, and in fact without which what we asked to prove is not necessarily true. Can you find it? (See page 38.)

0208 In the following addition, replace the letters by digits so that the calculation is correct. (All occurrences of a letter must be replaced by the same digit. Different letters stand for different digits.)

```
     J U N E
+  J O L L Y
+  H O L L Y
+      J O N
 -----------
 J O L Y O N
```

(The possibility of this puzzle was observed by a Hungarian schoolboy, a viewer of the fine B.B.C. television series, "The Forsyte Saga.")

Answer J = 1, H = 9, O = 0, L = 2, N = 5, Y = 3, E = 4, U = 7. The first 4 are easily obtained from considerations involving the 3 digits at the left hand end of the answer. From the 10's column we must have N = 4 or N = 5. The possibility, N = 4, forces Y = 7 and E = 6, or Y = 6 and E = 8, but then the 100's column gives U = 1 or 2 respectively, which are both ruled out since letters have already been assigned to these digits.

0209 An ordinary deck of cards is thoroughly shuffled and cut at random. Player A then exposes the cards from the top, one at a time, and for each card that is not an ace, B pays A 10 cents. However, as soon as an ace turns up, A pays B $1.20 and the game is finished. Is this game perfectly fair? If not, which player does it favour.

Answer In each game A pays B $1.20. He expects to receive $\sum_{r=1}^{49} P_r.(r-1).10$ cents, where P_r is the probability that the rth card in the deck is the first ace. [10(r–1) cents is the amount paid by B for a game ending on the rth card.] Now:–

$$P_r = {}^{48}C_{r-1}/{}^{52}C_{r-1} \,.\, 4/53-r$$

The factor ${}^{48}C_{r-1}/{}^{52}C_{r-1}$ is the probability that none of the first (r–1) cards are aces, the numerator being the number of such selections, and the denominator the total number of ways of selecting (r–1) cards from the pack. The factor 4/53–r is the probability that (after (r–1) non-aces) the rth card *is* an ace, since all 4 aces remain in a deck reduced to (53–r) cards. Hence the game favours A if

$$\sum_{r=1}^{49} \frac{{}^{48}C_{r-1}}{{}^{52}C_{r-1}} \cdot \frac{4}{(53-r)} \cdot 10(r-1) > 120.$$

At this stage, mathematical insight gives way to sheer calculation, which reveals that the sum of the series is very close to 96.

Thus, the game favours B.

Comment. To sum the series, Robert Kuhn programmed a computer (and so did I), and sent in the print out. Keith Burns found an ingenious method of handling the summation without machine assistance. See Problem 0216.

0210 *(Submitted by Malcolm Temperley, sometime Parabola solver, now at the Australian National University.)* Six houses on one side of a street are numbered from 20 to 30, from left to right. In each lives a man with a different occupation, pet, hobby and five of them smoke a different brand of cigarette. No two men are the same age and none is over sixty.

1. Rev. Martin who does not believe in playing sport on Sundays, lives in the house next to the church.
2. The banker ~~smokes Alpine~~.
3. The lawyer often complains about his neighbour's cat because of his prize pet pigeons.
4. Mr Clarke smokes Cambridge.
5. Mr Hughes is annoyed by his neighbour's loud music.
6. The 42-year-old man has a 30-year-old man to the left (to your left).
7. Dr Ferguson's neighbour is 8 years older than he.
8. Mr Sullivan trades coins *for* stamps with the man in number 28.
9. The 34-year-old man plays squash every day of the week.
10. Escort is smoked by the 53-year-old man.
11. Mr Walker lives next door to the Marlborough smoker.
12. The reporter, being the youngest, is not quite 30.
13. The executive lives in number 24.
14. Mr Hughes prefers pet goldfish to pet guinea pigs.
15. The budgies are often annoyed by the model planes next door.
16. Benson and Hedges are smoked in a house next to the house where the dog is kept.
17. The doctor lives in number 20.

Now who plays golf? Who does not smoke? And who keeps budgies? (P.S. *Medical authorities warn that smoking is a health hazard.*)

Answer

	20	22	24	26	28	30
NAME	Ferguson	Sullivan	Hughes	Clarke	Walker	Martin
JOB	Doctor	Lawyer	Executive	Reporter	Banker	Minister
AGE	45	53	34	29	30	42
HOBBY	Golf	Stamps	Squash	Music	Coins	Model Planes
SMOKE	Non-smoker	Escort	Benson & Hedges	Cambridge	Alpine	Marlborough
PET	Cat	Pigeons	Goldfish	Dog	Budgies	Guinea Pigs

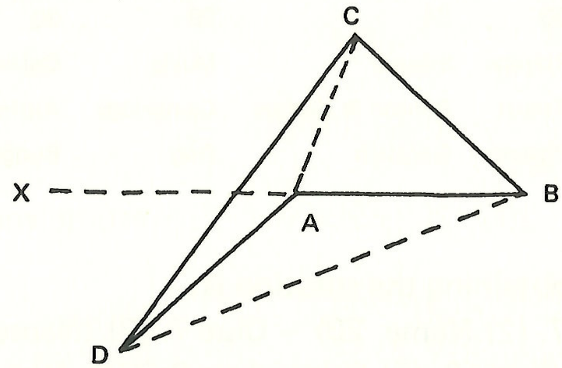
A reasonable order for obtaining the solution is:—
(1) (Job, 20) — Clue 17. (2) Name, 20) — Clue 7. (3) (Name, 30) and (Job, 30) — Clue 1. (4) (Job, 24) — Clue 13. (5) Using clues 6, 7, 9, 10 and 12 the ages 29, 30, 34, 42 and 53 are obtained. (6) (Age, 20) cannot be any of these — clues 6, 7, 12. (7) (Age, 20) = 53–8 and not 29–8, 30–8, 34–8, or 42–8 – again clues 6, 7, 12. Also (age, 22) = 53. (8) (Age, 28) and (Age, 30). Other possible places for the 30 and 42 age pair (clue 6) are easily ruled out, since the minister is neither 29 (clue 12), nor 34. (Clue 9). (9) (Age, 24) and (Age, 26). (10) (Hobby, 24) — Clue 9. (11) (Job, 26) — Clue 12. (12) (Smoke, 22) — Clue 10. (13) (Job, 28) and (Smoke, 28) — Clue 2. (14) (Job, 22) and (Pet, 22) — Clue 3. (15) (Hobby, 28) — Clue 8. (16) The trial Clarke — Cambridge in house 24, and Sullivan — stamps in 22 leads to music in 26 or 30, Hughes in 28, Walker in 26, contradicted by Clue 11. The trial Clark — Cambridge in house 24, Sullivan — stamps in 26, leads to Hughes — goldfish in 28, music in 30, Walker in 22, Marlborough in 20, B&H in 26 and dog in 24, hence cat in 20, and now it is impossible to satisfy clue 15. (17) Therefore: (Name, 26) = Clarke and (smoke, 26) = Cambridge (Clue 4). (18) (Name, 22) and (Hobby, 22) — Clue 8. (19) (Name, 28) and (smoke, 30) — Clue 11. (20) (Name, 24), (Hobby, 26) and (pet, 24) — Clues 5, 14. (21) (Pet, 20) — Clue 3. (22) (Smoke, 24) and (pet, 26) — Clue 16. (23) (Hobby, 30) and (pet, 28) — Clue 15.

Hence: Dr Ferguson plays golf and does not smoke, and Walker (the banker) keeps budgies. (Clue 14 also suggests that Rev. Martin's pets are guinea pigs. One would have felt more confident of this if the final question was "Who keeps Guinea pigs?" instead of "Who keeps budgies?".)

Answer to Comment on Problem O207

We should have specified that ABCD was a *convex* polygon. The "proof" given fails since the accompanying diagram is possible, in which BC = AD and ∠ABC = ∠DAX.

This was very easy to overlook, but I was delighted that R. Kuhn and K. Burns were equal to the occasion. — C. Cox.



Solvers of Problems 201—210 (Vol. 9 No. 1)

G. Cleary, I. Ritosa, J. Papageorgius, S. Stedman (South Sydney Boys' High) 201, 202.

J. Archibald, T. Hatziandreou, M. Lavidis (South Sydney Boys' High) 201.

R. Bong, S. Gardner, R. Goodey, R. Holman, P. Pavlov, P. Rodas, R. Sait, F. Venditti (South Sydney Boys' High) 202.

G. Berman (Sydney Grammar) 208, 210.

D. Bollinger (Newington) 201.

H. Burns (Campbell High, ACT) 203, 204.

K. Burns (Campbell High, ACT) 206, 207*, 209†, 210.

R. Casley (Gosford High) 206—208, 210.

B. Charlton (Wollongong High) 210.

J. Christodoulou (East Hills Boys' High) 208, 210.

S. Cogger (East Hills Boys' High) 203.

A. de Carvalho (St. Ignatius) 210.

P. Diacono (St. Joseph's) 208.

M. Diamond (Claremont, WA) 201—204.

V. Drastik (Sir Joseph Banks High) 206, 207.

A. Fekete (Sydney Grammar) 201—203, 205.

K. Hawtrey (James Ruse Agricultural High) 210.

P. Hehir (St. Ignatius) 203, 207, 208, 210.

J. Holten (East Hills Boys' High) 203, 208, 210.

D. Kleeman (Newcastle Boys' High) 208.
R. Kuhn (Sydney Grammar) 206, 207*, 208, 209†.
K. McGrade (Scholastica's College, Glebe) 208, 210.
E. Moon (Scholastica's College, Glebe) 208, 210.
M. Pearce (James Ruse Agricultural High) 203, 204.
G. Pendlebury (James Ruse Agricultural High) 210.
B. Quinn (St. Joseph's) 206–208, 210.
C. Quinn (Marist Brothers, Hamilton) 201, 202.
J. Schwartz (Cranbrook) 208.
A. Schwarzer (James Ruse Agricultural High) 208.
R. Sharp (Newington) 208.
A. Szeto (St. Joseph's) 206–208, 209 (an interesting answer, but not quite correct in detail).
N. Talent (Cheltenham Girls' High) 208, 210.
C. Watson (school not given) 208, 210.
R. Watson (James Ruse Agricultural High) 208, 210.
I. Wood (James Ruse Agricultural High) 201–202.
S. Wood (James Ruse Agricultural High) 203, 210.
L. Yager (Hornsby Girls' High) 208, 210.
R. Yager (Normanhurst High) 206–208, 209 (almost right), 210.

* The only two people who noticed the error in O207 were Keith Burns and Robert Kuhn. Both provided a faultless discussion. Their answer is given on page 38.

† Again in O209, Keith Burns and Robert Kuhn were the only two correct. Robert solved it by a computer, while Keith summed a series very similar to that mentioned in problem O216.

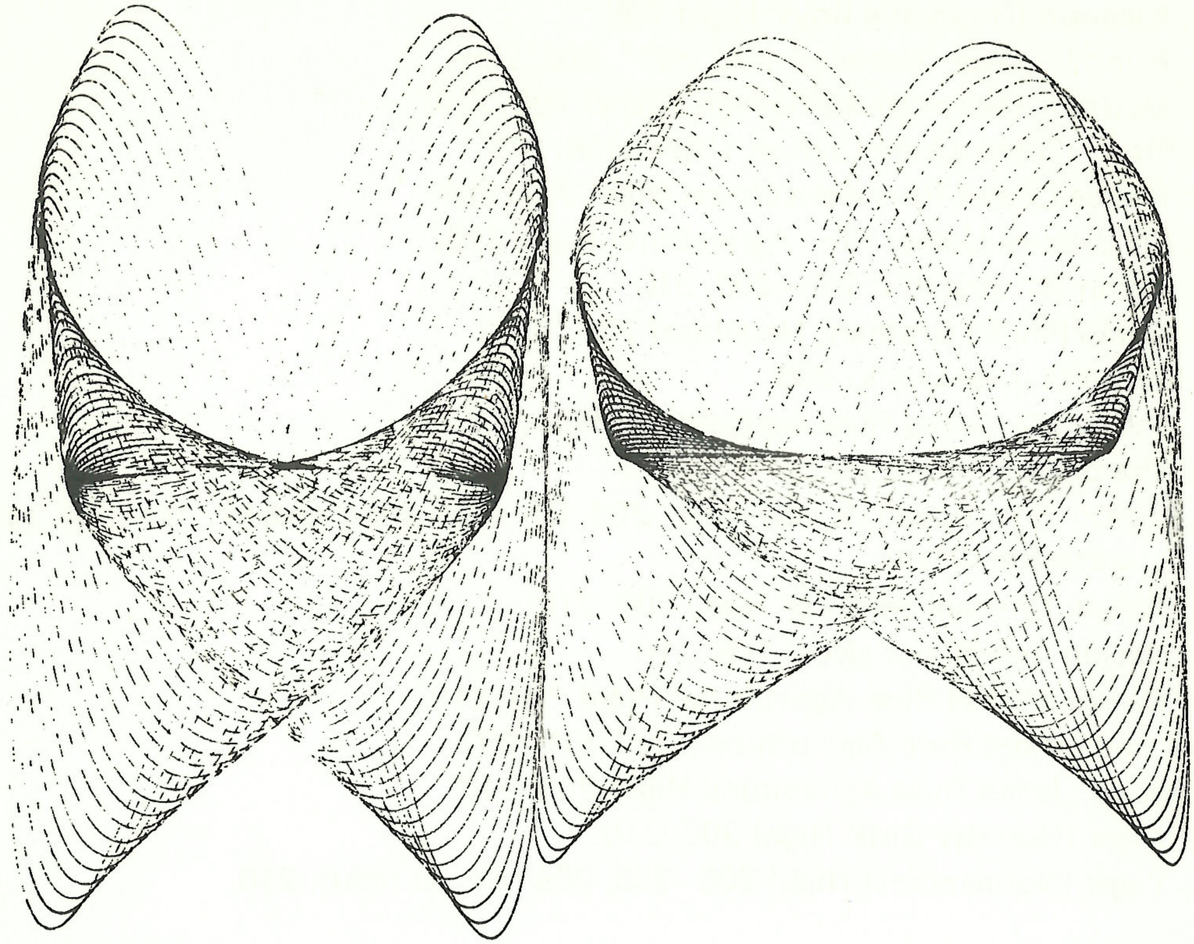
☆ ☆ ☆

An Intelligence Test Problem

Can you give the next term in the following sequence? (There *is* a trick.)

o, t, t, f, f, s, s,

Answer on page 40.



Answers

Problem Squares: If n had r digits, then n^2 would have 2r or 2r–1 digits. Since 100 is of the form 3r+1, the condition is impossible and so the probability is 0.

A Weather Report: The temperatures were –2, –1, 1, 2 and 3.

Squares and Cubes: $10^2 - 6^2 = 4^3, 10^3 - 6^3 = 28^2$.

Match Arithmetic: Arrange ten matches to form the letters F I V E. Three out of these ten matches form I V = 4.

An Intelligence Test Problem: one, two, three, four, five, six, seven, eight .