THE KLEIN BOTTLE

What has one side and no edges? This isn't an impossible riddle but has an answer viz. the Klein Bottle. This article will be divided into two topics dealing with the Klein bottle. Firstly, how the Klein bottle is related to the Moebius strip (see Vol. 8 No. 2) and other geometrical figures and secondly methods of construction of models of the Klein bottle. In all this it will be taken for granted that you realise the flexibility of form which is associated with topology.

[In topology, you are allowed to stretch and twist figures as much as you like. – Ed.]

1. How the Klein Bottle can be Arrived at

Using a flat square of paper and joining the edges in various ways, one arrives at a cylinder, a cone, a Moebius strip, a sphere, a torus, a projective plane and a Klein bottle.

The easiest way to demonstrate this is to use figures with arrows placed on the sides as follows:

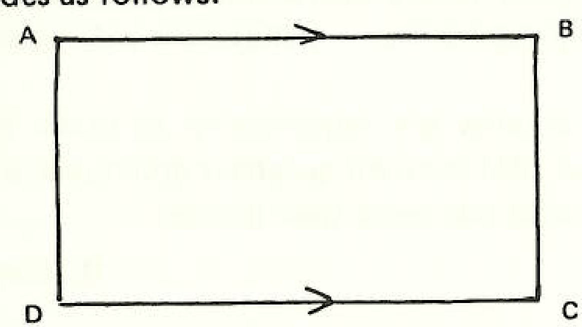

Figure 1

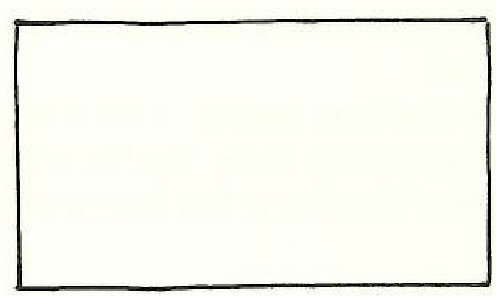

Figure 2

(a) Figure 1 indicates that the top edge (AB) and the bottom edge (CD) must be joined so that the arrows are pointing in the same direction i.e. AB joins CD with A at D and B at C.

(b) Taking a flat piece of paper as in Figure 2 we proceed to the cylinder, Moebius strip, and cone by joining two of the sides in different ways. With the top arrow pointing left to right, there are six different ways we can point an arrow on one of the three other sides (see Figure 3 over page).

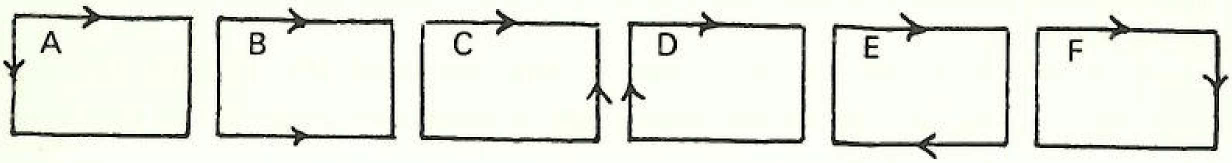

Figure 3

Now if one proceeds to construct Figure 3A i.e. moving the upper down to the left edge, an inverted cone with a corner in the lower edge is formed (see Figure 4). And this can be easily converted topologically to a normal cone. (Just one point, if one is to make these models with paper, a square is necessary so that all of each edge is joined.) When we construct Figure 3C we find it also is a cone. Thus Figure 3A and C (with arrows pointing away from or towards a common corner) are identical.

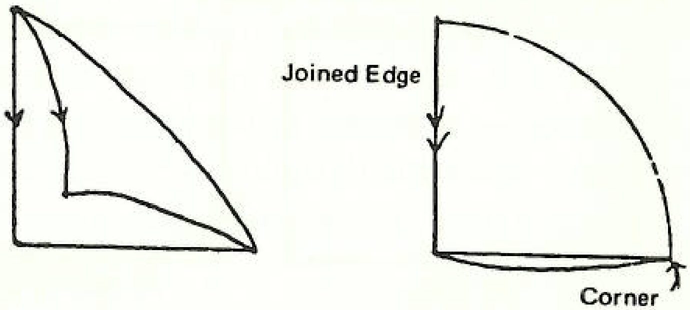

Figure 4

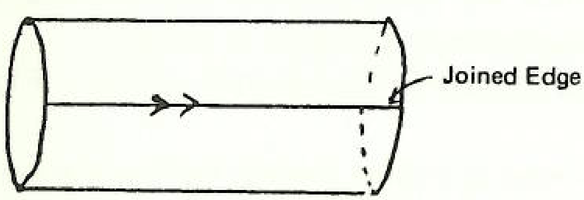

Figure 5

The construction of Figure 3B is the same as in Figure 1 and this of course forms the hollow cylinder of Figure 5.

Skip Figure 3D for a while and look at Figure 3E. It might seem almost impossible to join the top and bottom edges as it is until one does a little topological stretching and twisting. By lengthening the left and right hand sides and shortening the upper and lower edges, we have a long thin strip Figure 6(i). If we half twist the lower edge so that the arrow is pointing in the same direction as the top (Figure 6(ii)) we can then join these two edges easily (Figure 6(iii)). The result

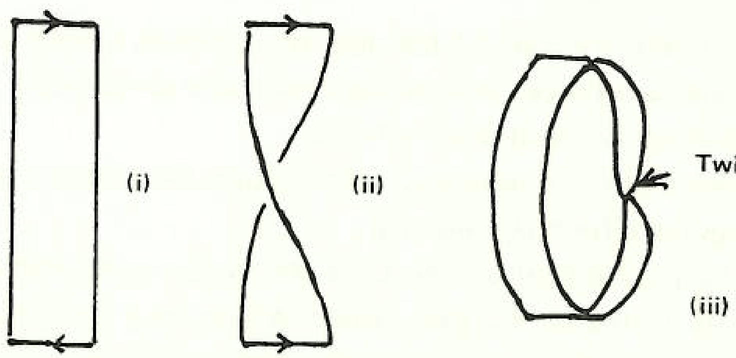

Figure 6

is the Moebius strip which has one side because the front has been joined to the back by the twist.

Back to Figure 3. We see that (D) is the same as (F) (except (F) has turned 90° clockwise). These are impossible to construct because the two sides are already joined at the wrong corner.

So, in summary, there are three basic figures formed by joining two sides. These are illustrated in Figure 3'.

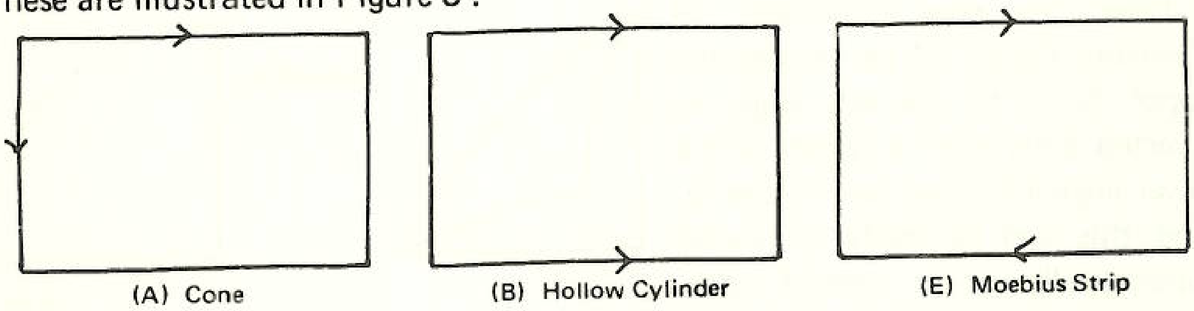

Figure 3'

Now what would occur if we joined *two* sets of two sides together? The two sets must be both opposite or both adjacent sides: for if one set were opposite and one set were adjacent the result would be that only three sides would be joined and not all four. (An interesting exercise would be to discover as many three-sides-joined figures as possible.) Thus we must perform the operations of Figure 3'B and E together or each of the operations of Figure 3' twice.

Suppose we do the operation of Figure 3'B twice. The result is shown in Figure 7.

Joining top and bottom we get a cylinder and then joining end to end we get the (doughnut shaped) torus.

Now let's do the operations of Figure 3'B and E. The diagram would be as shown in Figure 8. If you try this yourself you will find it rather difficult. It is this figure which is the Klein bottle whose construction will be more fully discussed later.

Finally the last of the sets of opposite sides is made by doing the operation of Figure 3'E twice. This is the even more awkwardly constructed "Projective plane" which we will not discuss here.

Now for the adjacent sides figure. By doing the operation of Figure 3'A twice we get a sphere (Figure 10).

In (i) the first set of joins forms the cone and when the next set are joined in (ii) it forms a "double cone" which may be distorted as in (iii) to a sphere (or cube, rectangular or triangular prism which are all topological identical).

Thus we have formed all possible combinations of two sets of two sides. The most interesting of all these is the Klein bottle which is a combination of a Moebius strip and a cylinder.

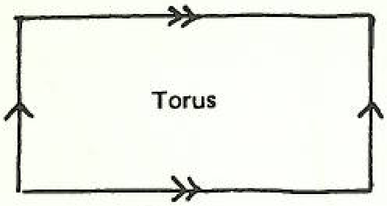

Figure 7

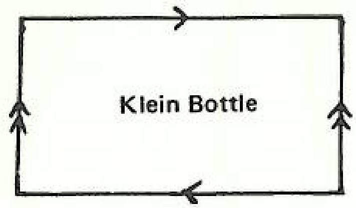

Figure 8

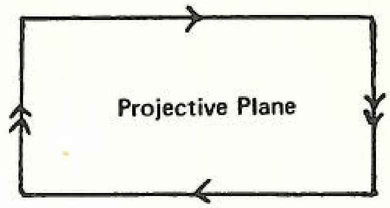

Figure 9

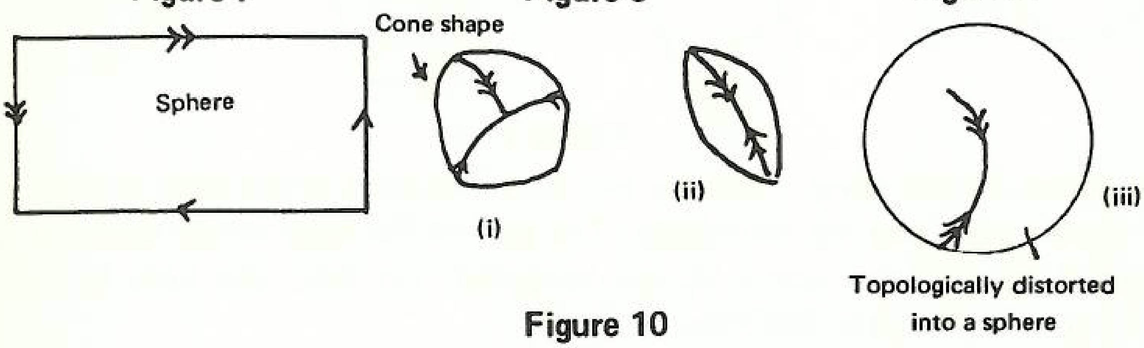

Figure 10

2. Construction of the Klein Bottle

But how can we construct the physical reality of the diagrammatic form of Figure 8?

If we start by joining the left and right sides together, forming a Moebius strip we run into trouble because the top and bottom edges merge into one another making it impossible to join one edge with itself (try it for yourself?). Thus the only way is to join the top and bottom in to a hollow cylinder and proceed from there. But when we try to join the ends as in Figure 7 we notice the arrows on either end are running in opposite directions. No matter how much we twist either edge around, they are *still* going in opposite directions! So the problem is how to get the edges going around in the same direction. If we turn the cylinder into a horseshoe (see Figure 11a), we notice that the edges are in the same direction. So we must merge these edges together from the inside and the other direction.

Starting with the horseshoe shaped cylinder (a), we topologically shorten and narrow one end and lengthen and widen the other (b). We then put the narrow end through a "hole in the side" of the large end (c) (the "hole" will be discussed later) and finally we join the two ends together on the inside (d): this is the usual model for the Klein bottle. One point to keep in mind: for the convenience of the model, a hole has to be formed, but mathematically there are no points of

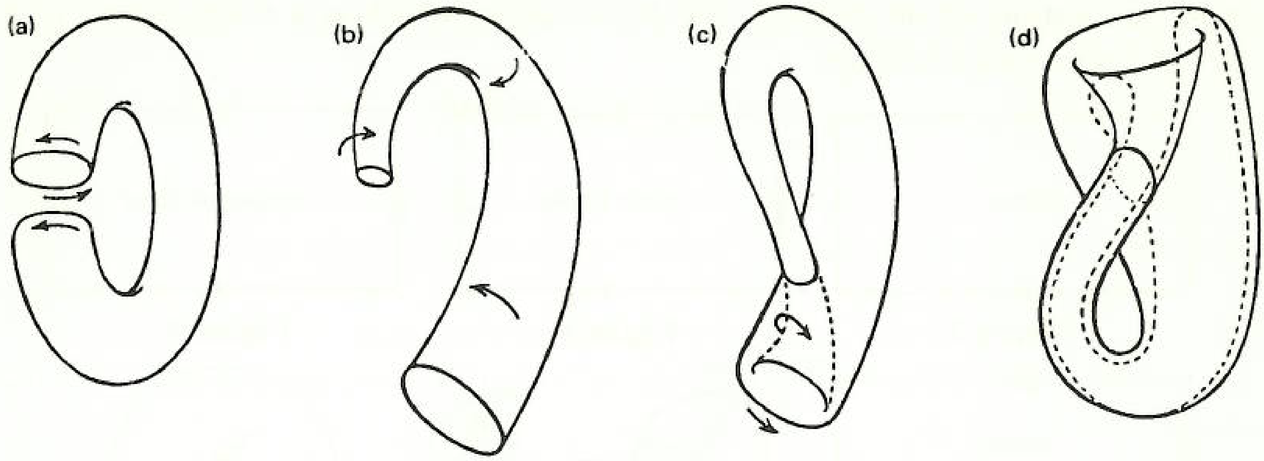

Figure 11

intersection on this curve. However two different parts of the body coincide, i.e. the narrow neck and the main body. The part of the body where the hole is, is supposed to continue across it uninterrupted. [In fact, this hole is, strictly speaking, not allowed in Topology. – Ed.]

To see this, try drawing a Moebius strip on a sheet of paper (from any perspective) and you will find that two of your lines must cross. This is the two dimensional equivalent of the "hole" in the bottle: in 4 dimensions, the spout does not have to enter the bottle through "a hole in the side".

One of the great properties of a Klein bottle is that it has only one side and no edges. The fact that the whole surface is continuous confirms the latter and if one follows the inside of the spout into the inside of the bottle, up to the top on the outside of the spout, along this to the outside of the bottle and back up again, one finds the inside of the spout again: this confirms the former. This was to be expected because in the construction diagram the vertical arrows of Figure 8 caused a twist which joined the front to the back and the horizontal arrows of Figure 8 caused the remaining edges to disappear.

Thus we have answered our original question and gone through most of the shapes common to geometry whilst doing so. This is what topology is basically about: travelling through the "rubber-sheet" mathematics jungle and emerging with a treasury of shapes and forms all related to one another.

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Editor's Note: Phillip has done a wonderful job. Maybe *you* would like to do a follow-up article on the projective plane (see Phillip's Figure 9) or one on what happens when the Klein bottle is cut in two (as was done last year for the Moebius strip). If you are interested and want some help let me know (stating the topic you are interested in) and I will send you some books to read.