HOW NOT TO CALCULATE AREAS

If asked to find the area bounded by the parabola $y = x^2$ the x-axis and the line $x = a$, you would write, almost instinctively, area = $\int_0^a x^2\,dx = \frac{1}{3}a^3$. It was not always this easy! Before the invention of the calculus in the mid 17th century, man had no such extensive tool available, and in his long struggle to find one he recorded both successes and failures. In the following pages the reader will find a few loosely connected historical interludes highlighting the nature of this struggle.

The first recorded attempts to find areas (and volumes) date back some 5000 years to the Babylonians. Many mathematical tablets, some dated at around 3000 B.C. have been deciphered and from these we can assert that the Babylonians 'knew' how to calculate the areas of rectangles and (at least isosceles) triangles. Of course they failed to produce the formula area = ab for the area of a rectangle of sides a and b, not only because they had no such notation, but more seriously, because they could not multiply. To the Babylonians multiplying a and b meant adding a to itself b times. Hence they experienced great difficulty if neither a nor b were integers. Nonetheless they did leave behind tablets containing example after example on the calculation of areas of a rectangle. In this sense they also knew the area of an (isosceles) triangle and of a circle. Their method for finding the area of a circle of radius r corresponds, in today's notation, to the formula: area $=(3.125)r^2$.

That the Egyptians also 'knew' the areas of rectangles and triangles may be inferred from the following problem in the Moscow Mathematical Papyrus. This important papyrus, discovered in 1890, dates from around 2000 B.C.

Free translation: Given a triangle (note the figure in amongst the hieroglyphics) of area 20 (appropriate square units) whose base in 2/5 its height, find both the base and height. Solution: Double the triangle to obtain a rectangle of area 40. Divide 2/5 into 1 to obtain 5/2. Multiply 5/2 by 40 to obtain 100. Take the square root of 100 to get height = 10, base = 4.

Further problems in the papyrus offer a method for calculating the area of a circle, viz, it is the area of a square of side 8/9 the diameter. In our notation this corresponds to the formula: area = $(3.16)r^2$. When one finds this formula being used by the government of the day to assess land tax it is worth noting that $3.16 > \pi$.

Until about 600 B.C. little advances beyond these primitive results were made. In the hands of the Greeks, mathematics as a whole evolved from a descriptive subject (as illustrated above) into a theoretical and deductive science containing precise statements (the theorems) complete with rigorous justifications (the proofs). Without a doubt the greatest mathematician of the latter stages of Greek mathematics was Archimedes (287–212 B.C.). Amongst many other results he is credited with the first *proof* that the area of a circle is proportional to the square of its diameter. The proof is based on the following lemma (attributed to Eudoxus); If a and b are any given positive numbers, one can find an integer n so large that $a/2^n < b$ (i.e., $2^{-n} \to 0$ as $n \to \infty$). Here then is a (quite free) translation of Archimedes' result:

Theorem: Let C_1 and C_2 be circles on diameters XY and AB. Then

$$\frac{\text{area } C_1}{\text{area } C_2} = \frac{(XY)^2}{(AB)^2} \quad \ldots (*)$$

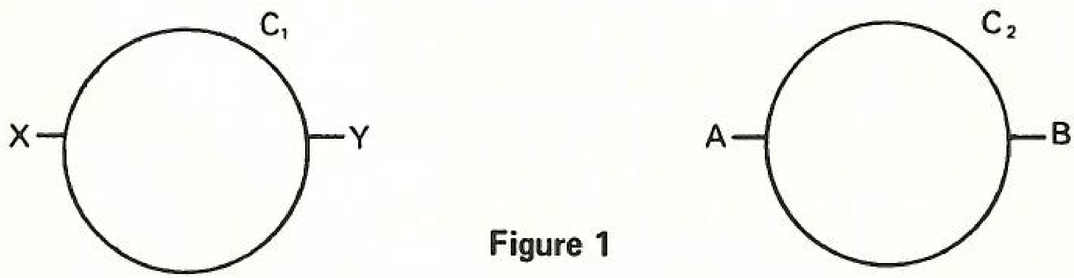

Figure 1

Proof: By way of establishing a contradiction suppose (*) is false. Then, for some region Σ with area $\Sigma \neq$ area C_2 we have

$$(XY)^2/(AB)^2 = \text{area } C_1/\text{area } \Sigma \ldots (1)$$

Suppose that in fact area $\Sigma <$ area C_2 (the case in which area $\Sigma >$ area C_2 can be handled similarly). Bisect the arcs $\widehat{AB}$ and $\widehat{BA}$ at the points C and D respectively. Inscribe the square ACBD and circumscribe the square PQRS (see Figure 2). Then area PQRS = 2 x area ACBD, so that area ACBD = ½ x area PQRS > ½ area C_2. Hence if we denote by A_1 the shaded region (i.e. $A_1 = C_2 -$ ACBD), we have

$$\text{area } A_1 < \tfrac{1}{2} \text{ area } C_2 \quad \ldots (2)$$

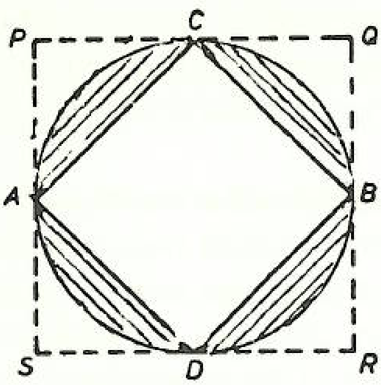

Figure 2

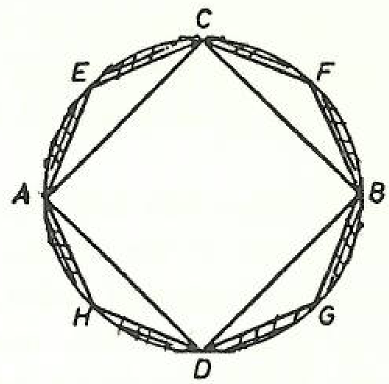

Figure 3

Next inscribe an octagon AECFBGDH (with E bisecting the arc $\widehat{AC}$ etc.) and denote by A_2 the shaded region between C_2 and the octagon (see Figure 3). Thus

$$\text{area } A_2 = \text{area } C_2 - \text{area } \Delta ACBD - \text{area } \Delta\, AEC - \text{area } \Delta\, CEB \qquad (3)$$
$$- \text{area } \Delta\, BGD - \text{area } \Delta\, DHA \ldots$$

However 2 area ΔAEC = area ACTU > area AEC (see Figure 4). Thus area $\Delta AEC > \frac{1}{2}$ area AEC, a similar result holding for the triangles CFB, BGD and DHA. Now area (AEC + CFB + BGD + DHA) = area A_1 so that area $(\Delta AEC + \Delta CFB + \Delta BGD + \Delta DHA) > \frac{1}{2}$ area A_1.

It follows then, from (2) and (3), that area $A_2 < 1/2^2$ area C_2.

Repeat this process finally arriving at a 2^n-gon Π_n inscribed in C_2 such that

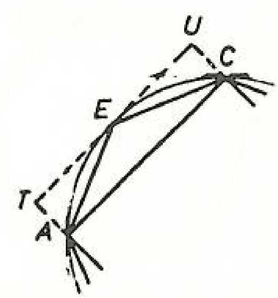

Figure 4

$$\text{area } C_2 - \text{area } \Pi_n < \frac{1}{2^n} \text{ area } C_2.$$

Then, by the principle of Eudoxus (with a = area C_2, b = area C_2 − area Σ) we can choose an n so large that area C_2 − area Π_n < area C_2 − area Σ. That is, area Π_n > area Σ . . . (4).

Now inscribe a similar polygon $\Pi_n{}'$ in C_1. By the known result that the ratio of the area of similar polygons = ratio of the squares on their diameters, and by (1),

$$\frac{(XY)^2}{(AB)^2} = \frac{\text{area } \Pi_n{}'}{\text{area } \Pi_n} = \frac{\text{area } C_1}{\text{area } \Sigma},$$

which implies that

$$\frac{\text{area }\Sigma}{\text{area }\Pi_n} = \frac{\text{area }C_1}{\text{area }\Pi_n{}'}, \qquad \ldots (5)$$

However, by (4), the left hand side of (5) is <1, whereas (because $\Pi_n{}'$ is inscribed in C_1) the right hand side of (5) is >1. Thus a contradiction has been reached, and so (*) holds. Q.E.D.

Using identical methods, Archimedes managed to *prove* the formulae for the area under a parabola and the surface area of a sphere, cone and cylinder. This technique of approximating the area under a curve by the areas of polygons is known as the method of exhaustion as it both exhausts the area under the curve and the person using the method. Via this exhaustive technique (to wit, by actually inscribing a 96-gon in a circle) Archimedes obtained the estimate for π: $3\frac{10}{71} < \pi < 3\frac{10}{70}$, i.e. $3.1408 < \pi < 3.1429$.

The method of exhaustion is open to the following criticism: by and large it does not help us find areas! To apply the technique one first must *guess* the right formula — the technique may then be used to verify the correctness of that guess. Thus the method is not a calculating tool — it is not yet a calculus. Nonetheless it contains the germinal idea behind the integral calculus, and certainly after the translation and printing of Archimedes' works in 1544 it exerted a profound influence on the evolution of the calculus.

It would however be a gross distortion of history to assert that the integral calculus developed directly from Archimedes to Newton/Leibnitz. Along the way a variety of different and ingenious attacks on the problems of area were made — many of these being stimulated by the pressing needs of commerce, navigation and astronomy. Of particular interest is an attack due to Cavalieri (1598 – 1647) — a professor at Bologna and a student and disciple of Galileo Galilei. His work entitled "The geometry of indivisibles" published in 1635 resurrected the Greek controversy regarding the nature of space, coming out strongly on the indivisible or atomic side. According to this view a line is made up of (a large number of) points, an area of (a large number of) lines etc. Thus to find the area of a triangle BC, Cavalieri considers it to be made up of n+1 lines containing 0, 1, 2, . . . , n points respectively. Similarly the rectangle ABCD consists of n+1 lines each containing n points (see Figure 5). Then

$$\frac{\text{area ABC}}{\text{area ABCD}} = \frac{\text{sum of the lines in ABC}}{\text{sum of the lines in ABCD}}$$

$$= \frac{0 + 1 + 2 + \ldots + n}{n + n + \ldots + n}$$

$$= \frac{\frac{1}{2}n(n+1)}{n(n+1)} = \frac{1}{2} \qquad !!!$$

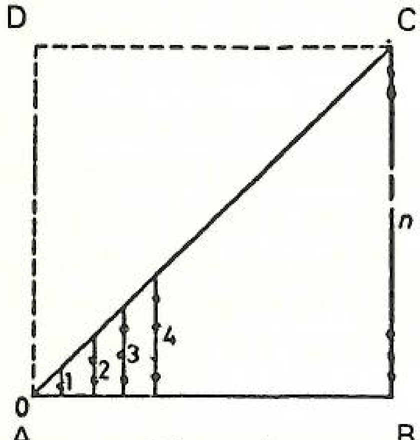

Figure 5

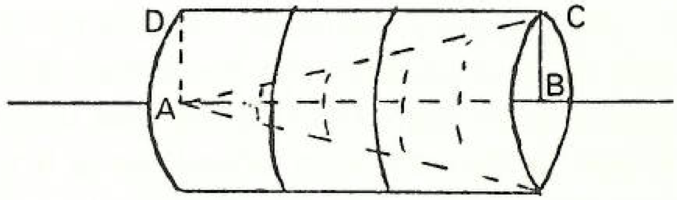

Figure 6

No doubt spurred on by this success, Cavalieri grew more ambitious and attempted to calculate the volume of a cone by rotating ABCD about the base AB as shown in Figure 6. Now if a line containing k points is rotated the resulting area will be proportional to k^2. Hence, as a volume consists of the sum of its cross sectional areas,

$$\frac{\text{Volume of Cone}}{\text{Volume of Cylinder}} = \frac{0^2 + 1^2 + 2^2 + \ldots n^2}{n^2 + n^2 + \ldots n^2}$$

$$= \frac{1/6[n(n+1)(2n+1)]}{n^2(n+1)} = \frac{1}{3} + \frac{1}{6n},$$

and (to paraphrase Cavalieri) as n is very large the 'error' $\frac{1}{6n}$ is very small and hence may be neglected to give the correct answer of $\frac{1}{3}$.

Cavalieri then guessed that (neglecting small errors) for any positive integer m,

$$\frac{0^m + 1^m + 2^m + \ldots + n^m}{n^m + n^m + n^m + \ldots + n^m} = \frac{1}{m+1} \qquad \ldots (6)$$

a result 'equivalent to' $\int_0 x^m \, dx = \frac{1}{m+1}$.

Torricelli, a student of Cavalieri, criticized this method of indivisibles with the following example:

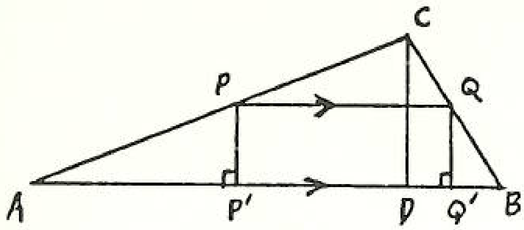

Figure 7

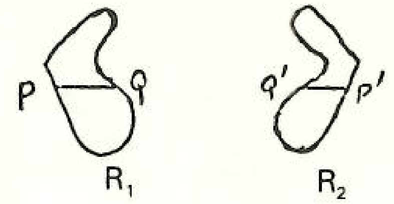

Figure 8

Let ABC be a *non*-isosceles triangle and for each point P on AC construct Q on BC so that PQ||AB, and denote by P′,Q′ the feet of the perpendiculars from P and Q respectively (see Figure 7). Then to each point P there corresponds precisely one point Q (and vice versa) and clearly PP′ = QQ′. Thus, adopting Cavalieri's approach,

$$\frac{\text{area ACD}}{\text{area BCD}} = \frac{\text{sum of the lengths of PP}'}{\text{sum of the lengths of QQ}'} = 1 \qquad !!!$$

Cavalieri appeared unconcerned by this criticism dismissing it with the reply that if, for example, BC consists of 100 points and AC of 200 points, there will be twice as many lines PP′ as there are lines QQ′ so the above ratio will not be 1.

In addition to the above examples, Cavalieri's book contains a quite general principle proved via the method of indivisibles.

Cavalieri's Principle: *If 2 regions have everywhere the same width they have the same area.*
i e. if R_1 and R_2 are such regions (see Figure 8), area R_1 = area R_2, as for each P, PQ = P′Q′.
(Problem: State and prove this result via the integral calculus.)

An excellent illustration of the use of this principle was provided in 1637 by the Frenchman Roberval. The (quite old) problem he solved was that of finding the area under (an arch of) the cycloid. A cycloid is the path described by a point on the circumference of a circle as that circle moves on a horizontal line, as shown in Figure 9.

Galileo had, some 50 years earlier, suggested that the area under the arch PBQ is 3 times that of the generating circle, i.e. area PBQ = $3\pi r^2$. Roberval confirmed this belief.

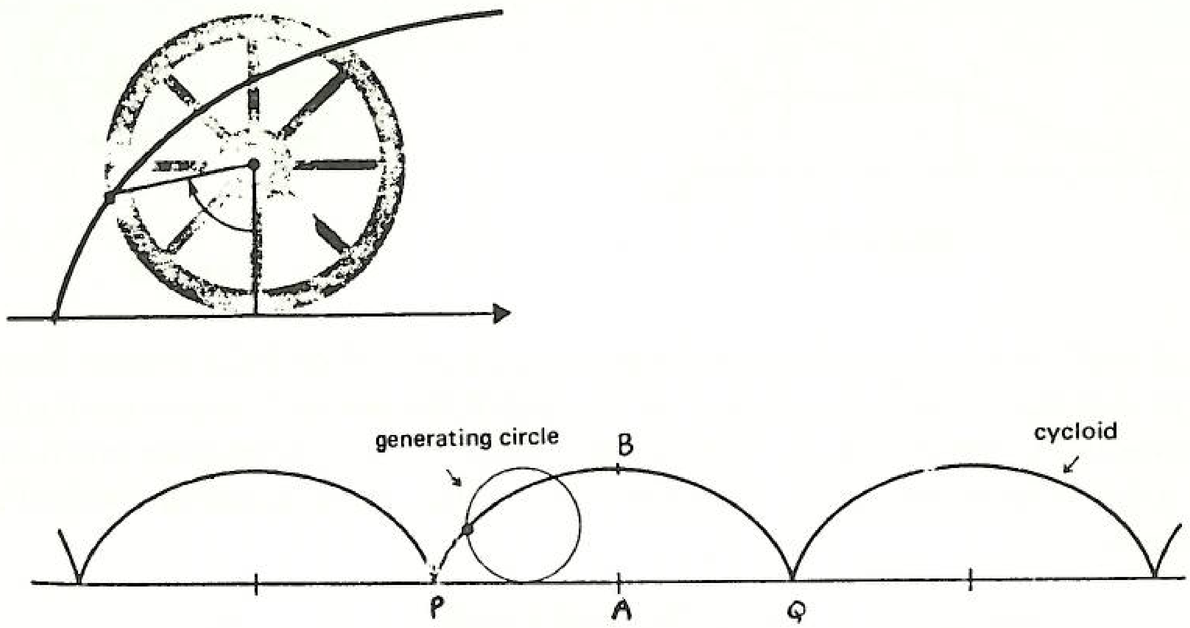

Figure 9

Consider one half PB of the arch, let A be the foot of the perpendicular from B, and for each point R on the cycloid construct S on the diameter CP so that SR||PA (see Figure 10). Denote by T the point on the semi-circle obtained by extending SR, and let V be that point on the extension of SR such that ST = RV. As R moves along the cycloid, V describes a curve C. (In fact C is a sine curve.) Then, for each line SV in PVBC there is a line S'V' in PVBA of equal length.

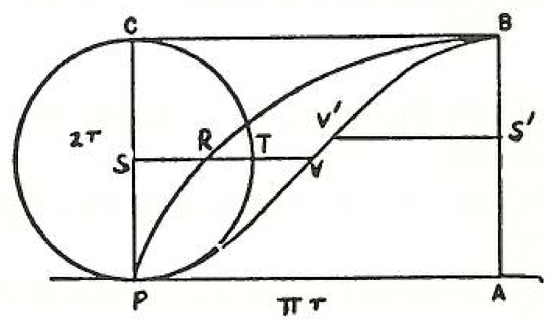

Figure 10

Hence, by Cavelieri's principle the curve C bisects the rectangle PABC into two equal areas. Thus

area PRBA = area PVBA + area PVBR
= ½ area PABC + area PVBR
= $\frac{1}{2}(\pi r \times 2r)$ + area PVBR
= area of generating circle + area PVBR.

Next, by the above construction, each line RV in PVBR corresponds to a line ST of equal length in the semi-circle $\widehat{PTC}$. So, again by Cavalieri's principle, area PVBR = area semi circle = ½ area generating circle. Therefore area PABR = 3/2 x area of generating circle and the area under an arch of the cycloid is 3 times that of the generating circle.

Such was the ingenuity needed to calculate simple areas. Via the same methods (i.e. by a combination of Cavelieri's principle and ingenuity), Pascal, in 1640, obtained certain areas which, in our notation, correspond to the evaluation of $\int_0^{\pi/2} \sin^n\theta \, d\theta$ for $n = 1,2,3 \ldots$. Wallis in 1655 'observed' that Cavalieri's earlier formula (6) holds for $m = ½$ so that (again in our language) $\int_0^1 \sqrt{x}\, dx = 2/3$. Wallis further 'found' that $\int_0^1 \sqrt{(1-x)^2}\, dx = \pi/4$.

At roughly the same time Cavalieri was developing methods based on the notion of indivisibles, Fermat began utilizing the counter-notion of infinitesimals – a concept fraught with as many difficulties as that of indivisibles. Fermat was clearly inspired by Archimedes' method of exhaustion, as his 'proof' that $\int_0^a x^{p/q} dx = a^{p/q+1}/p/q + 1$ shows.

To find the area under the curve $y = x^{p/q}$ (p and q positive integers) from $x = 0$ to $x = a$ first choose a real number r with $0 < r < 1$ and consider the situation in Figure 11.

To simplify the working write $A = a^{1/q}$ and $R = r^{1/q}$, and so $0 < R < 1$. Then the heights of successive rectangles (from right to left) are

$$A^p, \ (RA)^p, \ (R^2A)^p, \ldots$$

and their areas are

$$A^p(a-ra) = A^p(A^q - R^qA^q)$$
$$= A^{p+q}(1-R^q),$$
$$(RA)^p(ra-r^2a) = A^{p+q}(1-R^q)R^{p+q},$$
$$(R^2A)^p(r^2a-r^3a) = A^{p+q}(1-R^q)R^{2(p+q)}$$

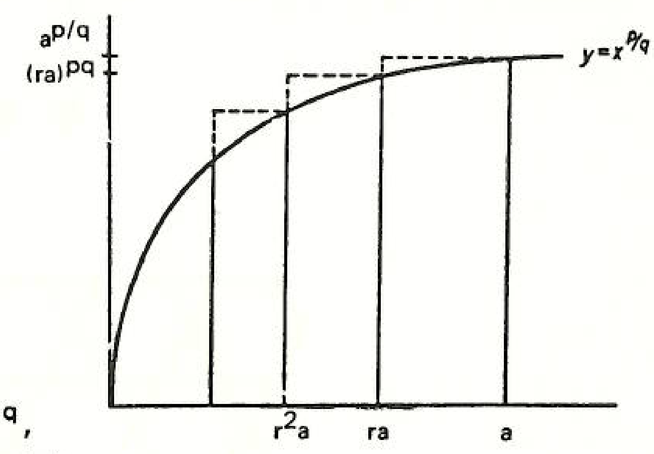

Figure 11

So the sum of the areas of all the rectangles is

$$A^{p+q}(1-R^q) \sum_{n=0}^{\infty} (R^{p+q})^n \qquad \ldots (7)$$

and as $R^{p+q} < 1$, this geometric series has sum

$$S = A^{p+q}(1-R^q) \times \frac{1}{1-R^{p+q}}$$

$$= A^{p+q} \times \frac{(1 + R + R^2 + \ldots + R^{q-1})(1-R)}{(1 + R + R^2 + \ldots + R^{p+q-1})(1-R)}$$

$$= A^{p+q} \times \frac{1 + R + R^2 + \ldots + R^{q-1}}{1 + R + R^2 + \ldots + R^{p+q-1}}$$

as $R \neq 1$.

The area under the curve can now be found by putting $r = 1$, so that the rectangles have zero width. (Note that if $r = 1$, $R = 1$ so both the last step above and expression (7) are inadmissable.) Then the area under the curve

$$= A^{p+q} \times \frac{1 + 1 + \ldots + 1 \text{ (q times)}}{1 + 1 \ldots 1 \text{ (p + q times)}} = \frac{q}{p+q} a^{(p+q)/q}$$

$$= \frac{a^{p/q+1}}{p/q+1}. \qquad \text{Q.E.D.}$$

Fermat's method was further developed by Isaac Barrow in his "Geometric lectures" of 1669. In particular, Barrow was able to compute $\int_0^a \tan x \, dx$. The scene was now set for Isaac Newton (a student of Barrow's) and Gottfried Leibnitz to simultaneously but independently discover the underlying principles of the calculus and to turn it into a calculating tool par excellence.

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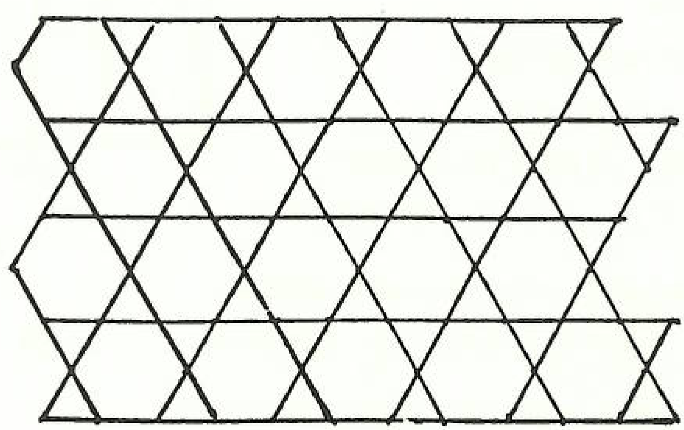

Tessellation