Open

O226 The houses on the same side of the street as Tom's house are numbered 1, 3, 5, . . . (with no odd number left out). The sum of the house numbers from Tom's to the end of the street is the same in both directions. If his house has a 3-digit number, what is it?

O227 Prove that for all positive integers, n,

$$\frac{\sqrt{2}}{2.\sqrt{(2n)}} \leqslant \frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots 2n} < \frac{\sqrt{3}}{2.\sqrt{(2n)}}.$$

O228 The radius of the inscribed circle of a triangle is 4 and the segments into which one side is divided by the point of contact are 6 and 8 units long. Find the lengths of the other two sides.

O229 Two identical circular cylinders with unit radius have axes which intersect at right angles. Find the volume of the region inside both cylinders.

O230 A game for two players involves a heap of matches. The first player, A, may pick up any positive number of matches, so long as at least one is left. Thereafter the players move alternately, a move consisting in picking up any positive number of matches not exceeding twice the number picked up by the opponent's move just completed. For example, if at some stage A picks up 3 matches, B may pick up 1, 2, 3, 4, 5, or 6 matches for his next play. The winner is the player who picks up the last match. If there are initially 60 matches in the heap, the first player can force a win. How?

SOLUTIONS

Solutions to Problems 211–220 in Vol. 9 No. 2.

Junior

J211 The numbers 31,767 and 34,924, when divided by a certain 3 digit divisor, leave the same remainder, also a 3-digit number. Find the remainder.

Answer: The difference of the two numbers must be a multiple of the divisor, since the remainders are equal. That is, the divisor is a 3-digit factor of 34,924–31,767 (which = 3,157 = 7 x 11 x 41) so must be either 7 x 41 or 11 x

41. Either of these leaves a remainder of 197 when divided into the given numbers.

Successful Solvers: A. Fekete (Sydney Grammar), R. Borg, D. Crawford, J. Dogan, P. Hatzi, G. Sheriff (all from South Sydney Boys' High).

J212 A painted wooden block has dimensions x cm., y cm. and z cm. where x, y and z are all whole numbers greater than 1. The block is sawn up by cuts parallel to the faces into 1 cm. cubes. How many of these are there with 3, 2, 1 and 0 painted faces respectively?

Answer: Only the 8 cubes occupying the corner positions have 3 painted faces.

Of the x cubes along an edge x cms long, the 2 end cubes have 3 painted faces, the other $x-2$ cubes have 2 painted faces. Since there are 4 edges x cms long, 4 edges y cms long, and 4 edges z cms long, there are altogether $4(x-2) + 4(y-2) + 4(z-2)$ cubes with 2 painted faces.

After the edge cubes have been removed from a face of the block, the remaining cubes in the face have 1 painted face, and occupy a rectangle 2 cms shorter and narrower than the original face. Hence, there are altogether $2(x-2)(y-2) + 2(y-2)(z-2) + 2(z-2)(x-2)$ cubes with 1 painted face.

The completely unpainted cubes occupy a rectangular region inside the block, of dimensions $x-2$ cms by $y-2$ cms by $z-2$ cms. Hence there are $(x-2)(y-2)(z-2)$ cubes with no painted face.

You can check that the sum of these expressions is identically equal to x.y.z, the total number of cubes in the block.

Successful Solvers: A. Fekete (Sydney Grammar), C. Quinn (Marist Brothers, Maitland); R. Borg (South Sydney Boys' High).

Intermediate

I213 Let P be a point inside a convex polygon, all of whose sides are of equal length. Prove that the sum of the lengths of the perpendiculars from P to the sides of the polygon (produced if necessary) is the same for all positions of P.

Is the converse true? Justify your answer.

Answer: Join P to all the vertices of the polygon, thus dissecting it into triangles. If the perpendicular distance from P to a side AB is h, the area of ΔABP is $\frac{1}{2}c.h$ where c is the length of a side of the polygon. Adding the areas of all the triangles we obtain

Area of polygon = ½c.Σh, where Σh is the sum of the perpendicular distances from P to all the sides. We can see that this sum Σh, being equal to 2 x Area of polygon/c, is independent of the position of P.

The converse is false. For example, the sum of the distances to the sides of a rectangle from an interior point is constant, but the sides are not all equal.

Successful Solvers: J. Burnett, K. Hawtrey (both from James Ruse Ag. High), A. Fekete (Sydney Grammar), P. Hehir (St. Ignatius).

1214 In how many different ways is it possible to express 2^n as the sum of four perfect squares?

Answer: Since the square of an odd number leaves the remainder 1 when divided by 8, the sum of 4 squares cannot be divisible by 8 unless all are even. Hence

$$a^2 + b^2 + c^2 + d^2 = 2^n \ (n \geqslant 3) \quad \text{if and only if } A^2 + B^2 + C^2 + D^2 = 2^{n-2},$$

where a = 2A, b = 2B, c = 2C, and d = 2D. Thus there are precisely the same number of ways of expressing 2^n and 2^{n-2} as the sum of 4 squares. Repeating the process, we see that if n ($\geqslant$ 4) is even, the number of expressions for 2^n as the sum of 4 squares is the same as for $2^2 = 4$; viz. two (4 + 0 + 0 + 0 and 1 + 1 + 1 + 1). If n is odd the number of expressions is the same as for 2; viz. one (1 + 1 + 0 + 0).

Successful Solvers: P. Hehir (St. Ignatius), D. Paterson (South Sydney Boys' High).

1215 If a and b are consecutive odd numbers (both positive) show that (a+b) is a factor of $a^b + b^a$.

Answer: Let a = 2n–1 and b = 2n + 1. Then

$$\begin{aligned} a^b + b^a &= (2n-1)^b + (2n+1)^a \\ &= (2n)^b + {}^bC_1(2n)^{b-1}(-1) + \ldots + (-1)^b + (2n)^a + {}^aC_1(2n)^{a-1}(+1) + \\ &\qquad \ldots + (+1)^a. \end{aligned}$$

The terms $(-1)^b$ and $(+1)^a$ cancel. All the remaining terms have 4n (= a + b) as a factor, except the second last terms of the two binomial expansions, viz. b.2n and a.2n, and their sum is clearly divisible by a + b.

O216 *(From Keith Burns' method of summing the series involved in O209)*

(i) Prove that $\sum_{k=1}^{n} k(k+1)(k+2)\ldots(k+a-1) = \frac{n(n+1)(n+2)\ldots(n+a)}{a+1}$

[For example, 1.2.3 + 2.3.4 + 3.4.5 + . . . + 23.24.25 = ¼.23.24.25.26]

(ii) Since it is fairly easy to see that any polynomial $f(x) \equiv a_0 + a_1x + \ldots + a_lx^l$ can be expressed as

$$b_0 + b_1x + b_2x(x+1) + \ldots b_lx(x+1)\ldots(x+l-1),$$

we obtain a method for summing $\sum_{k=1}^{n} f(k)$.

Illustrate by evaluating $\sum_{k=1}^{10}(k^4 + 6k^3 + 10k^2 + 5k - 1)$.

Answer (i): One straightforward method is by induction on n. Alternatively, observe that

$$k.(k+1)(k+2)\ldots(k+a-1) =$$

$$\frac{k.(k+1)\ldots(k+a) - (k-1).k.(k+1)\ldots(k+a-1)}{a+1}$$

Replace all terms of the sum using this identity, and observe that everything cancels out except the positive part of the last term, which is the required result.

Answer (ii): Since $x.(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x$ we have

$$x^4 + 6x^3 + 10x^2 + 5x - 1 = x.(x+1)(x+2)(x+3) - x.(x+1) - 1.$$

The sum required is therefore $\sum_{k=1}^{10} k.(k+1)(k+2)(k+3) - \sum_{k=1}^{10} k.(k+1) - \sum_{k=1}^{10} 1$, which, using (i),

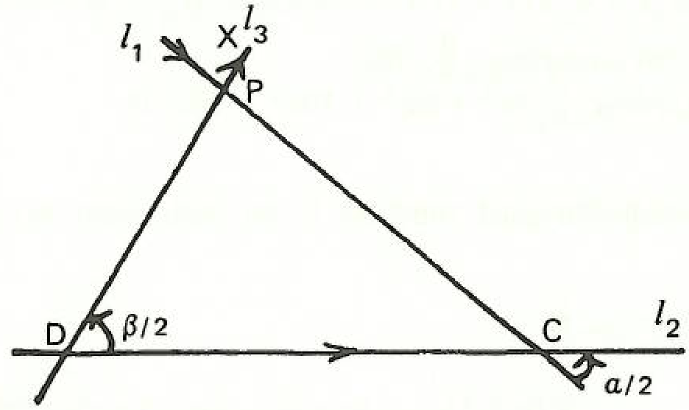
$$= \frac{10.11.12.13.14}{5} - \frac{10.11.12}{3} - 10 = 47{,}598.$$

Successful Solvers: V. Drastik (Sir Joseph Banks High), M. Ellison (Sydney Boys' High), P. Hehir (St. Ignatius), A. Oliviero (Newington), D. Powers (Fort Street Boys' High), B. Quinn (St. Josephs), J. Schwartz (Cranbrook).

O217 Let $T_{\alpha,C}$ and $T_{\beta,D}$ be two rotations about different points C,D with α,β both acute angles (see article on Geometry done with mirrors). Find the centre of the rotation $T_{\alpha,C}T_{\beta,D}$. How do you know it *is* a rotation?

Answer: As mentioned in the article Geometry done with Mirrors, Vol. 9 No. 2, if l_1 and l_2 are any 2 lines inclined at an angle ½a meeting at C, then $T_{a,C} = S_{l_1} S_{l_2}$ (The sense of the rotation is from l_1 to l_2). Choose for l_2 the line DC, then take for l_1 the line PC with ∠PCD = ½a, and for l_3 the line PD with ∠PDC = ½β. Then $T_{a,C} \cdot T_{\beta,D} = (S_{l_1} . S_{l_2}).(S_{l_2} . S_{l_3}) = S_{l_1} . S_{l_3}$ since $S_{l_2} . S_{l_2}$ is the identity transformation. Since l_1 and l_3 meet at P, $S_{l_1} . S_{l_3}$ is the rotation about P through twice the angle ∠CPX i.e., through $a + \beta$.

Successful Solvers: S. Barker (Condell Park High) – incomplete, V. Drastik (Sir Joseph Banks High), B. Quinn (St. Joseph's).



O218 A tortoise is observed by a number of people over a period of six minutes. Each person watches it for exactly one minute, and observes that it has travelled 10 feet in that minute. There is no instant during the 6 minute period that the tortoise is not observed by at least one person. Yet it travels a total of 100 feet during the 6 minutes. Explain how this is possible.

Answer: One solution is as follows. In the first 12 seconds the tortoise travels 10 feet. After standing still for 48 seconds, he again travels 10 feet in 12 seconds. One observer watches for the first minute, a second starts watching after 12 seconds and stops after 1 min. 12 secs. Hence after 1 min. 12 secs. the tortoise has travelled 20 feet. This pattern is repeated 5 times altogether.

Comment. In stating the problem, I should have made it clear that each observer watched the tortoise *continuously* for one minute. Then 100 feet is the greatest distance the tortoise could have travelled. If the observations do not have to be continuous the tortoise can stand still for the first minute, then travel 100 feet or any other distance, however large, over the next 5 minutes. A sufficiently large number of observers all watch him for nearly all the first minute, then each looks away except for some brief period later while the tortoise travels 10 feet. A number of solvers found this sort of solution.

Successful Solvers: J. Burnett (James Ruse Ag. High), D. Paterson (South Sydney Boys' High), B. Quinn (St. Joseph's), J. Schwartz (Cranbrook), P. Thomas (Hurlstone Ag. High).

Solutions allowing interrupted observation: V. Drastik (Sir Joseph Banks High), M. Ellison (Sydney Boys' High), P. Hehir (St. Ignatius), J. Holten (East Hills Boys' High), A. Oliviero (Newington), D. Powers (Fort Street Boys' High), A. Suurvali (Hurlstone Ag. High).

O219 A rectangular floor is l cm. long and w cm. wide. It is exactly covered with tiles each 1 cm. wide and c cm. long (where c, l and w are all whole numbers). Prove that either l or w is divisible by c.

2	3			c	1	2	3			c	1	2	3		
1	2	3			c	1	2	3			c	1	2	3	
c	1	2	3			c	1	2	3			c	1	2	3
	c	1	2	3			c	1	2	3			c	1	2
		c	1	2	3			c	1	2	3			c	1
3			c	1	2	3			c	1	2	3			c
2	3			c	1	2	3			c	1	2	3		
1	2	3			c	1	2	3			c	1	2	3	

Answer: Imagine the floor divided into 1 cm by 1 cm squares which are coloured using c different colours as indicated in the figure by the numbers 1, 2, 3, . . ., c. It is clear that each laid tile exactly covers c of these squares, one of each colour, so the tiling is impossible unless there are equal numbers of tiles of each colour.

Let $w = q.c + r$, where $0 \leqslant r < c$. Since in the figure the q.c rows of squares at the bottom of the diagram clearly contain equal numbers of squares of each colour, the tiling is possible only if the same is true of the upper r rows. Similarly, if $l = Q.c + R$, where $0 \leqslant R < c$, we may delete all save the R columns of squares at, say, the right of the figure. Thus a necessary condition for the tiling to be possible is that there are equal numbers of squares of each of the c colours in a rectangular block r cms by R cms. For definiteness, suppose $r \leqslant R$. Then there are diagonals in the rectangular block containing r squares all similarly coloured. To contain equal numbers of squares with all c colours, the block would need to contain at least c.r squares. It actually contains r.R squares, a smaller number unless r = 0.

Successful Solvers: M. Ellison (Sydney Boys' High), D. Paterson (South Sydney Boys' High).

O220 The ancient Outer Mongolian unit of length was the Mong. Its length in modern units is still the subject of investigation by a friend, an archaeologist, but it is known that one Mong is greater than one tenth of a metre and less than one metre. The matter may be cleared up if it is possible to understand the ancient notation employed for large numbers, since there has been unearthed a stone giving in Mongs the distance from Ulan Bator to Xanadu. Let us call it X Mongs. My friend, who is, regrettably, a compulsive gambler, has made a wager with me that when X is known in ordinary decimal notation the first digit will prove to be less than 5. (If he wins I give him $1, if he loses he gives me $1). I believed at first that since there were only 4 digits available to him (1, 2, 3 or 4) but 5 to me (5, 6, 7, 8 or 9) that the odds favoured me. Is this the right way of considering the question? What odds would you consider fair?

Answer: You will be familiar with the method widely used by scientists of denoting any number by the product of a number between 1 and 10 and a power of 10. For example, $7{,}536 = 7.536 \times 10^3$. Now suppose a large number, N, of arbitrarily selected distances are measured in Mongs, expressed in scientific notation, and the power of 10 then discarded. Let the number of results lying between x and $x + \delta x$ (where δx is very small) be $n(x,\delta x)$. We denote $n(x,\delta x)/N.\delta x$ by $P(x)$, which might be called the distribution density function of the collection. By its definition, $P(x).\delta x$ is the probability that if one of the results in the collection is picked out of a hat, it will be found to lie between x and $x+\delta x$, (provided δx is very small). The probability, $P(a,b)$, that the selected result lies in an interval (a,b) can be calculated by partitioning the interval into very small sub-intervals of length δx, and adding the probabilities for these. Thus $P(a,b) = \int_a^b P(x)\,dx$, where we have used the calculus notation for this summation process. In particular, the probability that its first digit is less than 5 would be $\int_1^5 P(x)\,dx$. Note that $\int_1^{10} P(x)\,dx$ must equal 1.

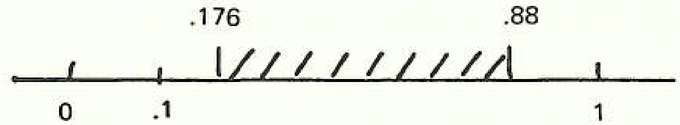
Well, this gives us the formal machinery for calculating the required odds, if only the function P(x) appropriate to the question were known. But how is it to be found? Even if we were prepared to carry through experimentally the process by which P(x) was defined, we should not be able to do so, since the length of the Mong is unknown. Strangely, our very ignorance may be used to suggest the suitable formula for P(x). Suppose we had been asked to bet on the probability that the first digit in the number of Mongs in the *return* trip was less than 5. Then what would be the correct odds? Surely they must be exactly the same, since we lack any information which would enable us to distinguish between the two cases.

What this boils down to is that if all the distances in our suitable sample of N were doubled, or increased by any other factor k, exactly the same distribution density function should result. The $P(1)\ \delta x$ of these which previously yielded results between 1 and $1 + \delta x$, will yield results between k and $k + k.\delta x$ after all distances are increased by the factor k ($1 \leqslant k < 10$). Since this interval previously contained $P(k).(k\delta x)$ results, for the distribution to be unchanged we must have $P(1) = P(k).k$ for all k in $1 \leqslant k < 10$. That is, P(x) is the function c/x where c is the constant P(1).

Since $\int_1^a c/x.dx = c.\log_e a$ (a standard result from calculus), the requirement $\int_1^{10} P(x).dx = 1$ yields $c = 1/\log_e 10$, and then $\int_1^5 P(x).dx = c.\log_e 5 = (\log_e 5/\log_e 10 = \log_{10} 5 = .6990$. Thus this argument from ignorance suggests that the odds favouring the occurrence of a first digit less than 5 are .6990 to .3010, or about 2.32 to 1.

Does that argument convince you? I would be more satisfied with it if the distance from Ulan Bator to Xanadu was quite unknown. But this is contradicted by the wording of the problem since the length of a Mong would not then be deducible by deciphering X. Before dismissing the argument too hastily, it may be of some interest to point out that the logarithmic distribution it predicts is actually *observed* to a reasonable degree of accuracy if lists of measured quantities are examined. Nevertheless, the argument depends entirely on the assumption that we have no other way of assessing the odds, and therefore no means of distinguishing between the odds for the given distance and any other. Let us consider another approach.

First, find the distance from Ulan Bator to Xanadu in kilometres. (This will require some research on the location of Xanadu. With a great deal more research one might be able to arrive at an estimate of the ancient road distance between the two towns, which would almost certainly be the distance on the unearthed stone). By way of example, suppose the result of our research enables us to place a figure of 880 kilometres on this distance. Then if 1 Mong = .88 metres, the number X is a power of 10. The first digit of X is less than 5 if 1 Mong lies between .88/5 = .176 metres and .88 metres; otherwise it is 5 or larger. What is the probability that the length of 1 Mong should fall in this shaded part of the interval (.1,1); (See Figure) The answer depends on the probability function for the length of the Mong, which we do not know. (If it is F(x), then the probability



that the length of the Mong is in the shaded region is $\int_{.176}^{.88} F(x).dx.$)

We can proceed further only by making a guess (supported, doubtless, by some sort of argument) as to the function F(x). One simple-minded try would take F(x) = constant on the assumption that it is equally likely for the length of the Mong to be in segments of the number axis of equal length. This does not seem to me to be a reasonable guess. It is simply not true for our society that there are just as many ordinarily used units of length between say, 0 yards and 1 yard as there are between 1 mile and 1 mile + 1 yard.

A far more reasonable guess would be that there is an equal likelihood for the Mong to be in any two intervals (a,b) and (c,d) if b/a = d/c. In fact, if the corresponding function F(x) is found, it yields the same result as was obtained by our first approach (viz. odds of log 5 : log 2) irrespective of the actual distance betwen U.B. and X. However, as pointed out cogently in the answer submitted by M. Ellison, it is highly likely that the units of length used by the ancient Mongolians is some naturally occurring distance, e.g. the length of one foot, or one arm, or one pace of the average ancient citizen.

Anyhow, it is clear that, if our first argument is rejected, we have insufficient information to arrive at "correct odds" for the wager. Any reasonable argument you come up with which suggests what *you* would regard as fair odds (which is, intentionally, how we phrased the question) is of interest.

PS: Assuming the problem attracted public attention and widespread gambling on the outcome, do you know how a bookmaker would arrive at the odds *he* was prepared to give? This interesting question must be left for discussion in a later Parabola.

[Any comments? – Ed.]

Successful Solvers: V. Drastik (Sir Joseph Banks High), M. Ellison (Sydney Boys' High), D. Powers (Fort Street Boys' High), J. Schwartz (Cranbrook).