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AN APPROACH TO THE SOLUTION OF MATHEMATICAL PROBLEMS

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Many of the articles in Parabola pose problems which invite the reader to invent original solutions. But, in the absence of a helpful hint, how does one set about arriving at such solutions?

We could go on indefinitely describing various ways of approaching mathematical puzzles and in the end we might even come to believe that there are as many ways of grappling with specific mathematical questions as there are problems . . . But undoubtedly the common denominator of most methods is hard work! Assuming that we have a certain "mathematical ability" let us consider the following question: "Does hard work alone guarantee us a solution?".

It is believed by some mathematicians and psychologists that there is another factor which is just as important to the solution of a problem as the unavoidable hard, conscientious work and "racking the brain". This factor has been called the "incubation period" and many mathematicians have found that hard work, in conjunction with this incubation period, has led to the production of original, elegant and successful solutions.

"Incubation period" may be defined as a time when one is not consciously working on the problem at hand – for instance it might be a time of relaxation involving listening to music, travelling or playing sport – and the peculiar beneficial effect that such a period of intellectual rest appears to have, has prompted several investigators to try to unravel the mystery.

For example in 1945 Jacques Hadamard had two ideas about why the incubation period + hard work combination was so successful. The first one, the "rest-hypothesis", supposed that the occurrence of fruitful ideas is often prevented by brain fatigue which only a period of *rest* can cure.

The second interpretation, the "forgetting-hypothesis", stressed that the incubation period could remedy the adverse effects of many of the factors which actually hinder creative insights. These might be, for example, the traps which are laid by too much conscious faith in traditional methods and the blind following of false leads and unfruitful leads. In these cases the incubation period offers an opportunity during which the mind can simply *forget* erroneous tacks, having once explored them, and develop unorthodox approaches to problems unfettered by the limitations usually imposed on the conscious mind by authority and tradition.

Hadamard put it this way:

> It can be admitted that an essential cause of illumination may be the *absence of interferences* which block progress during the preparation stage. "When, as must often happen, the thinker

makes a false start, he slides insensibly into a groove and may not be able to escape at the moment . . . Incubation would consist in getting rid of false leads and hampering assumptions so as to approach the problem with an 'open mind'." We can call this the *forgetting-hypothesis.*

The investigations of Hadamard and others definitely indicate that solutions of great elegance and simplicity may be overlooked if the mathematician does not stop a while to "incubate". To support this view Hadamard enumerated several of his own failures. One, which he particularly regretted, was a classic case of using familiar techniques and following false leads too strictly.

> It concerns the celebrated Dirichlet problem which I, for years, tried to solve in the same initial direction as Fredholm did . . . But physical interpretation, which is in general a very sure guide and had been most often such for me, misled me in that case. It suggested an attempt to solve the problem by a "potential of simple layer" — in that question, a blind alley — while the solution was to be looked for in the introduction of a "potential of double layer." This shows how . . . one ought not to follow too stubbornly a . . . principle, however justifiable and fruitful in general.

Although mathematicians generally agree that the incubation period is a time of rest there is no unanimity about what actually happens during this rest period. Henri Poincaré certainly had some definite thoughts on the matter which were based on his personal experiences of the fruitfulness of the incubation period.

Let us look closely at his account and interpretation of the mental processes which led to the successful establishment of a certain class of mathematical functions called Fuchsian functions. In his view the incubation period was much more than just a rest period since it was filled with the most active *subconscious* work the fruits of which filtered through into the conscious mind in the form of "ideas".

Poincaré's account begins with a description of what happened one evening some fifteen days after he had been working steadily on the problem without success. He drank a cup of black coffee and settled back into his chair. Many ideas began to pass through his mind until, in a general way, he was able to establish the existence of a class of Fuchsian functions. The immediate task facing him then was to ". . . represent these functions by the quotient of two series . . ." which he soon accomplished without difficulty. The following day he left his home town of Caen to join a geological excursion organised by the local School of Mines and the business of travelling made him forget his mathematical work but when he put his foot on the step of the bus he experienced a sudden flash of inspiration. He realised clearly that the transformations he had used to help define the Fuchsian functions were identical with those of non-Euclidean geometry and he verified this idea when he returned home.

The problem was still not completely solved and he tried repeatedly to finish the details but without success. Somewhat disgusted with his failure he decided to abandon the problem altogether and have a holiday. At the seaside he tried to think of something else. Then it happened again. One morning whilst walking on the cliffs an idea came to him ". . . with just the same characteristics of brevity, suddenness and immediate certainty . . ." as on the previous occasion. It became clear to him that the arithmetic transformations of indeterminate ternary quadratic forms were also identical with those of non-Euclidean geometry. When he arrived back from holidays he successfully deduced a number of necessary consequences of this insight but to his dismay one crucial aspect of the solution was missing. Unable to solve this remaining difficulty he left for the mountains where he was preoccupied with the demands of military service. Again the solution came to him suddenly whilst he was crossing the boulevard.

> I did not try to go deep into it immediately, and only after my service did I again take up the question. I had all the elements and had only to arrange them and put them together. So I wrote out my final memoir at a single stroke and without difficulty.

After all these similar and impressive experiences Poincaré became convinced of the incontestable importance of unconscious work in mathematical invention.

> These sudden inspirations . . . never happen except after some days of voluntary effort which has appeared absolutely fruitless and whence nothing good seems to have come, where the way taken seems totally astray. These efforts then have not been as sterile as one thinks; they have set agoing the unconscious machine and without them it would not have moved and would have produced nothing.

In fact as well as raising some interesting questions in psychology Poincaré also foreshadowed one side of the modern debate about whether or not all the multifarious functions of the mind will, in time, be reduced to the programmed operations of a computer.

> Now we have seen that mathematical work is not simply mechanical, that it could not be done by a machine, however perfect. It is not merely a question of applying rules, of making the most combinations possible according to certain fixed laws. The combinations so obtained would be exceedingly numerous, useless and cumbersome. The true work of the inventor consists in choosing among these combinations so as to eliminate the useless ones or rather to avoid the trouble of making them, and the rules which must guide this choice are extremely fine and delicate. It is almost impossible to state them precisely; they are felt rather than formulated.

Let us now return to the main theme of this paper. Whatever we believe we know about what actually happens when our conscious minds are "at rest" and our ideas are "incubating", the conjunction of hard work and a period of unconscious incubation has resulted in a fund of good ideas in mathematics in the past. Perhaps we should consciously try to put this two-part combination into practice when we are confronted by a troublesome problem. Of course this procedure presents major difficulties given the current system of examining students of mathematics. This consideration makes one wonder whether some very gifted and potentially original minds are being penalised because they are not able to make full use of this historically tried and tested approach to the solution of mathematical problems at those very moments when they are being assessed and their futures determined.

To end on a lighter note we might consider the advice of another creative individual, the novelist Constance Robertson, who has found that her best ideas occur in an environment of peaceful mountains with their sun-drenched valleys. It would seem that, taking all this advice into account, we should all assiduously campaign for take-home exams, long weekends and extended vacations on the grounds that at least some of us are being prevented by lack of leisure and suitably inspiring scenery from having fundamental insights into the world of mathematics.

Suggested Reading:

1. J. Hadamard, *An Essay on the Psychology of Invention in the Mathematical Field,* Princeton: Princeton University Press, 1945.
2. H. Poincaré, *The Foundations of Science,* Lancaster, Pa.: The Science Press, 1913.

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DID YOU KNOW

That the largest prime number yet found is $2^{19937}-1$? This prime was found in April, 1971 at the IBM Research Centre in New York. The IBM 360/91 computer there took 39.44 minutes to check that it was a prime number!

See the article on page 2.

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