YOUR LETTERS

Dear Sir,

In Problem O230 I noticed that those positions where the second player can force a win are all Fibonacci numbers. Since then I have been trying to find the reason for this and I decided to investigate what happened when each player could take n times as many as the previous player. For n = 1 the positions for the second player's win are powers of two. For n = 3 the series is (to 60): 1, 2, 3, 4, 6, 8, 11, 15, 21, 29, 40, 55. 1, 2 and 8 are both in the series for n = 1, 2, 3 while 3, 21, 55 are in n = 2, 3. For n = 4: 1, 2, 3, 4, 5, 7, 9, 12, 16, 20, 25, 32, 41, 53. I wrote this program to calculate these series (the program is in basic):

```
10  INPUT N
20  PRINT "N="N
30  DIM A[60]
40  A[1]=1
50  PRINT "1"
60  FOR I=2 TO 59
70  FOR J=1 TO (I-1)
80  R=I-J
90  IF J>= (A[R]/N) THEN 120
100 A[I]=J
110 GOTO 150
120 NEXT J
130 A[I]=I
140 PRINT I
150 NEXT I
160 END
```

The interesting fact about the series is that the difference between any consecutive numbers in any of them is also a member of the series. Perhaps some other reader has an explanation for this.

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[Alan seems to have misread the question: his statement 90 suggests that a player may not pick up N times the number picked up by his opponent, whereas the

question states that he may not pick up *more* than that number. If you are trying to follow Alan's programme, write $T_1 = 1$ and, from then on, the (k+1)th term T_{k+1} is the next A_i after T_k which is larger than T_k. I believe that the formula is $T_{k+1} = T_k + T$ where T is the smallest term in the series with $T \geqslant T_k/N$.
Maybe someone might like to improve Alan's programme – Ed.]

Dear Sir,

The method usually taught for summing the series $\sum_{k=1}^{n} k^r$ depends on knowing $\sum_{k=1}^{n} k^{r-1}$. However, using the method of problem 0216 in Vol. 9 No. 3, we can sum the above series with no information except the combinatorial coefficients ${}^zC_y$ for positive integers z,y. We set

$$x^r = a_0 + a_1x + a_2x(x+1) + \ldots + a_rx(x+1)\ldots(x+r-1) \qquad (1)$$

and try to determine the coefficients $a_0, a_1, \ldots a_r$. This may be done by setting x = 0, –1, –2, . . . –r in equation (1).

$$\text{Then } \sum_{x=1}^{n} x^r = a_0 + a_1\sum_{x=1}^{n} x + a_2\sum_{x=1}^{n} x(x+1)\ldots + a_r\sum_{x=1}^{n} x(x+1)\ldots(x+r-1)$$

$$= a_1 \cdot \tfrac{n}{2}(n+1) + a_2 \cdot \tfrac{n}{3}(n+1)(n+2) + \ldots + a_r \cdot \tfrac{n}{r+1}(n+1)(n+2)\ldots(n+r)$$

$$= \sum_{j=1}^{r} a_j \cdot \tfrac{n}{j+1}(n+1)\ldots(n+j)$$

where $a_j = \frac{1}{(j-1)!} \sum_{i=1}^{j} {}^{j-1}C_{i-1}(-1)^{i-1} \cdot (-i)^{r-1}$.

Although this calculation becomes somewhat lengthy when r is large, the advantages can be seen if we say let n = 1,000,000. This method also enables us to find the following limit.

$$\lim_{n\to\infty} \sum_{x=1}^{n} x^r \Big/ n^{r+1} = \lim_{n\to\infty} a_r \cdot \frac{n(n+1)\ldots(n+r)}{(r+n)n^{r+1}}$$ since other numerators have fewer terms in n.

$$= \lim_{n\to\infty} \frac{a_r}{r+1}$$

$$= \frac{1}{r+1}$$ since $a_r = 1$ (equating coefficients of x^r).

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Dear Sir,

NUMBER ANALOGIES WITH THE UNIVERSE

We have a problem in discussing the universe. In the ultimate, the limit is our brain's finite capacity but a much more immediate limit is our educational background. The world around is is, in the main, finite and we are taught to count and add finites: for example 3 + 4 = 7.

Even in more advanced mathematics we do not know that $\frac{1}{\infty} = 0$: we can only extrapolate our results as the denominator becomes larger. If we use the laws of arithmetic and cross-multiply we get $1 = \infty \times 0$ where there is no convergence since $n \times 0 = 0$ for all integers n. What do we do here?

Nevertheless, I think we can use number concepts of infinity to discuss an infinite universe. There is often heard in discussions on the universe that an infinite universe means an infinite number of suns just like ours an infinite number earths and an infinite number of you's and me's.

This is not the case. Consider the infinite set of integers: even though this set is infinite each member of it has unique characteristics and is considerably distinct from other members.

This does not mean that the bodies in the universe (if it is infinite) are necessarily unique, but it does mean that we cannot exclude that possibility.

On the other hand if we do find another planet the same as Earth (a very hypothetical situation), are there then an infinite number of earths? Are the only alternatives one and infinity? Considering the set of integers again there is only one integer 13 yet there are an infinite number of powers of 13 that share many of the characteristics of 13. The question is: how close is the similarity between the "Earths" (they are not the same object — they do not occupy the same positions in space)? Is there the same correlation as between the powers of 13?

This requires a close analysis of how similar 13, 13^2, 13^3, . . . really are. Perhaps someone else has some ideas?

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