SCHOOLS MATHEMATICS COMPETITION

Junior Division

1. Determine all ordered triples (a,b,c) of natural numbers, $a \geqslant b \geqslant c$, such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

Prove that you have found all solutions.

Answer: Clearly, a, b or c = 1 is impossible, so $a \geqslant b \geqslant c$ and $0 < \frac{1}{a} \leqslant \frac{1}{b} \leqslant \frac{1}{c} < 1$.

(i) c = 2. Then $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$ for a solution. As b = 2 is impossible, $\frac{1}{a} \leqslant \frac{1}{b} < \frac{1}{2}$.

(a) b = 3. Then a = 6 is the only solution.

(b) b = 4. Then a = 4 is the only solution.

(c) b > 4. Then $\frac{1}{a} + \frac{1}{b} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. No solution.

(ii) c = 3. Then $\frac{1}{a} + \frac{1}{b} = \frac{2}{3}$ for a solution.

(a) b = 3. Then a = 3 is the only solution.

(b) b > 3. Then $\frac{1}{a} + \frac{1}{b} < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. No solution.

(iii) c > 3. Then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$. No solution.

2. If $p = 13$ then $2p + 1$ is the cube of an integer. Prove that for no other prime number p is this true.

Answer: $2p + 1 = x^3$ gives $2p = x^3 - 1 = (x-1)(x^2 + x + 1)$. As $x^2 + x = x(x + 1)$ is always even for $x > 0$, $x^2 + x + 1$ is always odd, so that $x - 1 = 2$ or $2p$. But $x^2 + x + 1 = 1$ is impossible so $x - 1 = 2$, $x = 3$, and $p = 13$.

3. In how many different ways is it possible to pay one dollar in 1, 2 and 5 cent coins? Generally, in how many different ways is it possible to pay n dollars in 1, 2 and 5 cent coins?

For instance, 10 cents can be paid in 10 different ways, namely:

$$\begin{array}{c} 1+1+1+1+1+1+1+1+1+1 \\ 1+1+1+1+1+1+1+1+2 \\ 1+1+1+1+1+1+2+2 \\ 1+1+1+1+2+2+2 \\ 1+1+2+2+2+2 \\ 2+2+2+2+2 \\ 1+1+1+1+1+5 \\ 1+1+1+2+5 \\ 1+2+2+5 \\ 5+5 \end{array}$$

Answer: If the number of 1 cent pieces is x, the number of 2 cent pieces is y and the number of 5 cent pieces is z, we are to solve:—

(i) $x + 2y + 5z = 100$ and

(ii) $x + 2y + 5z = 100n$.

(i) (a) When z is even, x is even so we put $x = 2t$ and $z = 2u$ and solve (iii) $t + y + 5u = 50$ for $u = 0, 1, 2, \ldots, 10$.

That is, we have to solve $t + y = 5v$ for $v = 10, 9, 8, \ldots, 2, 1, 0$.

Now the number of solutions of $t + y = m$ is $m + 1$ as $0 \leqslant t \leqslant m$ and $y = m-t$.

So the number of solutions for (i) is

$$(50+1) + (45+1) + \ldots (5+1) + (0+1) = 11 + (50 + 45 + \ldots + 5)$$
$$= 11 + (50 + 45 + \ldots + 5)$$
$$= 11 + \frac{55}{2} \times 10$$
$$= 286.$$

(b) When z is odd, we have to solve (iv) $x + 2y = 5(20-z) = 5w$ where $w = 1, 3, 5, \ldots, 19$ (ten numbers in all).

The number of solutions of $t + 2y = m$, m odd, is equal to the number of values possible for y and this is $\frac{m+1}{2}$, as $y = 0, 1, 2, \ldots, \frac{m-1}{2}$.

So the number of solutions possible for (iv) is

$$\left(\frac{5+1}{2}\right) + \left(\frac{15+1}{2}\right) + \left(\frac{25+1}{2}\right) + \ldots + \left(\frac{95+1}{2}\right) = 3 + 8 + 13 + \ldots + 48$$
$$= \frac{(3+48)}{2} \times 10$$
$$= 255.$$

Answer to (i): $286 + 255 = 541$.

(ii) (a) Similar reasoning leads to the number of solutions for z even as:—

$$(50n + 1) + [(50n-5) + 1] + [(50n-10) + 1] + \ldots + (5 + 1) + (0 + 1)$$
$$= (10n + 1) + \frac{(50n+5)}{2} \times 10n$$
$$= (10n + 1) + 250n^2 + 25n$$
$$= 250n^2 + 35n + 1$$

Check: $n = 1$ gives 286.

(ii) (b) Again we get, similarly, number of solutions with z odd as:—

$$\left(\frac{5+1}{2}\right) + \left(\frac{15+1}{2}\right) + \left(\frac{25+1}{2}\right) + \ldots + \frac{(100-5)+1}{2}$$

$$= 3 + 8 + 13 + \ldots + 50n - 2$$

$$= \frac{50n+1}{2} \times 10n$$

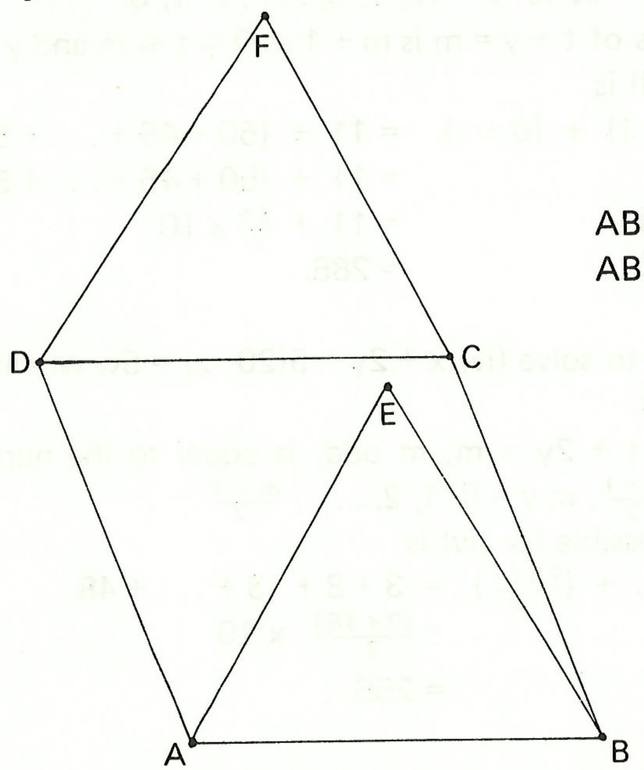
$$= 250n^2 + 5n$$

Check: $n = 1$ gives 255.

Answer to (ii): $500n^2 + 40n + 1$.

4. It is possible to select three points in the plane so that each of the points is at distance 1 from the other two. (Take the vertices of an equilateral triangle with side 1.)

Find six points in the plane so that each point is at distance 1 from exactly three of the other five points.



ABCD is any rhombus.
ABE, DCF are equilateral triangles.

5. Four persons with first names Alexander, Barry, Charles and David have this same set of names as their surnames. The following are known:

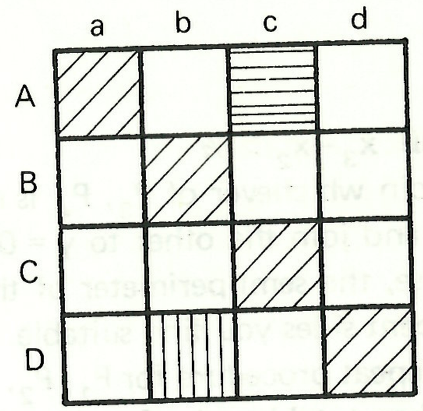
(i) No person has identical first and second names.

(ii) The surname of Charles is not Alexander.

(iii) The surname of Barry is identical with the first name of that person whose surname is the first name of the person with surname David.

What are the full names of the four persons?

Answer: a,b,c,d are first names. A,B,C,D are surnames. The shaded squares represent combinations of names that cannot occur from the data up to (ii). (iii) tells us that we have bY, yZ, zD.



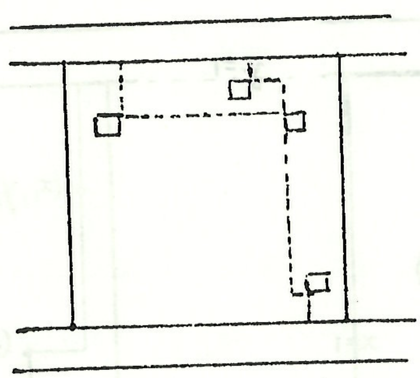
(iii) and the diagram shows that we cannot have bC, cB nor bC, cD so we must have bA. So we have aZ, zD with Z as B or C. As bD is impossible, we have aC, cD. Thus the names are:–

Alexander Charles, Charles David, David Barry, Barry Alexander.

6. There are two parallel highways running along the sides of a square paddock of width 1 km. There are four houses on the paddock and we want to construct footpaths running parallel to the sides of the paddock so that the occupants of the houses can walk to both highways on these footpaths.

Prove that this can always be accomplished so that the total length of the paths is not more than $2\frac{1}{2}$ km. Find an arrangement of the houses so that it is impossible to construct the required footpaths with total length less than $2\frac{1}{2}$ km.

NOTE: Pedestrians are not allowed to walk on the highways.



Answer: Let the roads be $y = 0$, $y = 1$ and the houses by P_1, P_2, P_3, P_4 at (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) where $0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant 1$. (This does NOT necessarily mean $y_1 \leqslant y_2$ etc!).

(i) $x_3 - x_2 \leqslant \frac{1}{2}$

Draw the line $x = x_3$ from $y = 0$ to $y = 1$. Connect P_1, P_2, P_4 to it by drawing as much of $y = y_1$, $y = y_2$, $y = y_4$ as is necessary to do this.

Then the total length of paths $= 1 + (x_4 - x_3) + (x_3 - x_2) + (x_3 - x_1)$

$= 1 + (x_4 - x_1) + (x_3 - x_2)$
$\leqslant 1 + 1 + ½$
$= 2½.$

(ii) $x_3 - x_2 > ½$

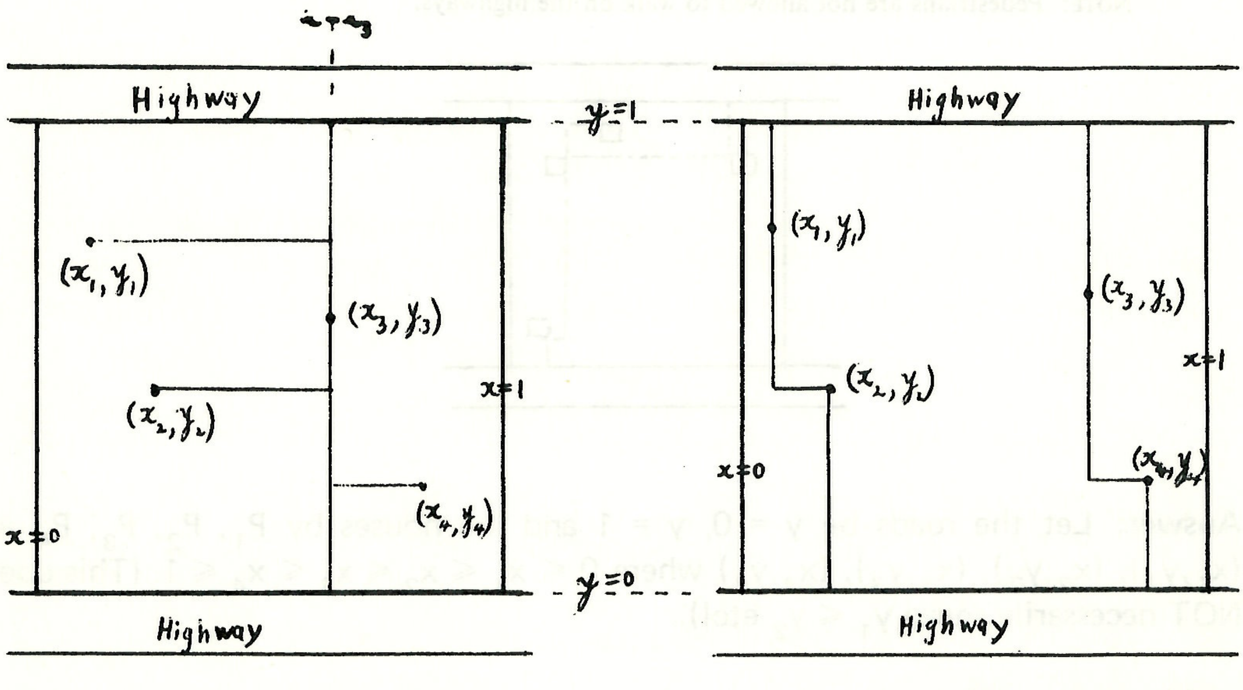
Join whichever of P_3, P_4 is nearest to $y = 1$ to it by the appropriate line $\parallel$ to y axis and join the other to $y = 0$ in the same way. The remainder of the path is, of course, the semi perimeter of the rectangle based on P_3, P_4 — whichever pair of adjacent sides you find suitable.

Repeat procedure for P_1, P_2.

Then total length of paths $= 1 + 1 + (x_4 - x_3) + (x_2 - x_1)$
$< 2½$

as $(x_4 - x_3) + (x_2 - x_1) + (x_3 - x_2) \leqslant 1$.

One arrangement for which 2½ km. is the minimum total length of paths is: $x_1 = 0$, $x_2 = ¼$, $x_3 = ¾$, $x_4 = 1$ and $y_1 = ¾$, $y_2 = 0$, $y_3 = ¼$, $y_4 = 1$.



(i) $x_3 - x_2 \leqslant \frac{1}{2}$ km (ii) $x_3 - x_2 > \frac{1}{2}$ km

Senior Division

1. Three real numbers u,v,w satisfy

$$0 < u < 1; \quad 0 < v < 1, \quad 0 < w < 1.$$

Prove that at least one of the three numbers

$$u(1-v), \quad v(1-w), \quad w(1-u),$$

is smaller than or equal to $\frac{1}{4}$. Assuming also that

$$u \leqslant v \leqslant w,$$

prove that at most one of them is greater than $\frac{1}{4}$.

Answer: (a): The product of the three numbers is

$$u(1-v) \,.\, v(1-w) \,.\, w(1-u) = u(1-u) \,.\, v(1-v) \,.\, w(1-w).$$

But $u(1-u) = u-u^2$ has a maximum of ¼ at u = ½. Hence the product on the right hand side is at most $(¼)^3$, and therefore at least one of the three factors on the left hand side is smaller or equal to ¼.

(b) If $u \leqslant v$, then $u(1-v) \leqslant u(1-u) \leqslant$ ¼ again. Similarly $v(1-w) \leqslant$ ¼ for $v \leqslant w$ and so at least two of the factors are smaller than or equal to ¼.

2. (i) The polynomial $a_0 + a_1x + a_2x^2$ has integer coefficients and its value is divisible by 3 for all integers x. Show that a_0, a_1 and a_2 are all multiples of 3.

(ii) $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ has integer coefficients and its value is divisible by 5 for all integers x. Show that all five coefficients are divisible by 5.

(iii) Construct a polynomial of degree 5 with integer coefficients not all divisible by 5 whose value is always divisible by 5 when x is an integer.

Answer: (1) Substitution of 0, 1 and 2 for x shows that a_0, $a_0 + a_1 + a_2$ and $a_0 + 2a_1 + 4a_2$ are all divisible by 3. So, therefore, are

$$a_1 = (a_0 + 2a_1 + 4a_2) - (a_0 + a_1 + a_2) - 3a_2$$

and $$a_2 = 2(a_0 + a_1 + a_2) - (a_0 + 2a_1 + 4a_2) + 3a_2 - a_0.$$

(ii) The method given here generalises to show that, for any prime p, a polynomial f(x) such that f(0), f(1), . . . , f(p−1) are all divisible by p has all its coefficients integers divisible by p. (Is this true for composite numbers, e.g. p = 4? No, as is shown by the example $f(x) = 2x + 2x^2$.)

Let $f_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

Division by x−4 yields the expression (with remainder)

$$f_4(x) = \beta_4 + (x-4)f_3(x)$$

where $\beta_4 = f_4(4)$ is divisible by 5; $f_3(x)$ is a cubic polynomial and $f_3(0)$, $f_3(1)$, $f_3(2)$ and $f_3(3)$ are all divisible by 5.

Likewise we have

$$f_3(x) = \beta_3 + (x-3)f_2(x)$$

where $f_2(x)$ is a quadratic polynomial and β_3, $f_2(0)$, $f_2(1)$ and $f_2(2)$ are all divisible by 5.

Likewise we have

$$f_2(x) = \beta_2 + (x-2)f_1(x),$$

where $f_1(x)$ is a linear polynomial and β_2, $f_1(0)$ and $f_1(1)$ are all divisible by 5.

Likewise we have

$$f_1(x) = \beta_1 + (x-1)\beta_0$$

where β_1 and the constant polynomial β_0 are both divisible by 5. Thus

$$f(x) = \beta_4 + \beta_3(x-4) + \beta_2(x-4)(x-3) + \beta_1(x-4)(x-3)(x-2) + \beta_0(x-4)(x-3)(x-2)(x-1)$$

where all the β's are divisible by 5. Hence so are all the coefficients of f.

(iii) A suitable example is $f(x) = (x-4)(x-3)(x-2)(x-1)x$.

3. Given n weights w_i, $i = 1,2, \ldots, n$ such that $2^{i-1} \leqslant w_i < 2^i$, show that every weight less than 2^n can be determined by the use of only these given weights. (All weights are assumed to be integers.)

Answer: The stated result is true for $n = 1$ (when a single unit weight is available). For $n = 2$, the available weights are 1 and 2, or 1 and 3. In the latter case 2 is represented as $3-1$. The problem lends itself to being solved by induction. So assume the result for $n = k-1$.

If the weight w to be determined is such that $w < 2^{k-1}$, the weight w_k is not needed to determine it.

Otherwise, $2^{k-1} \leqslant w < 2^k$. As $2^{k-1} \leqslant w_k < 2^k$, $-2^{k-1} < w-w_k < 2^{k-1}$. Hence $|w-w_k|$ can be determined by $w_1, \ldots, w_{k-1}$. Hence w can be represented by $w_1, \ldots, w_k$. This completes the induction.

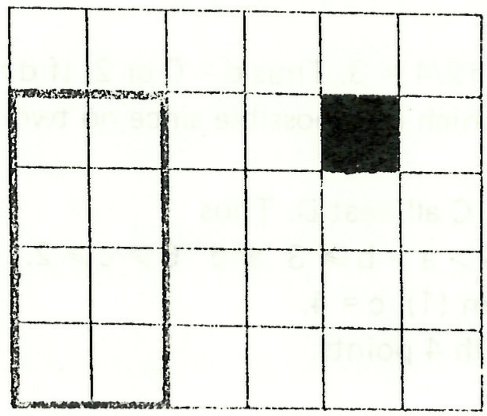
4. The diagram shows an $m \times n$ rectangle (with $m = 6, n = 5$) dissected into unit squares. Choosing the lower side and the left-hand side of the rectangle as co-ordinate axes, each square will be referred to by giving the co-ordinates of its upper right-hand corner. For example, the filled in square in the diagram is the (5,4) square.

By the "(2,4) box" is meant the region enclosed by the heavier lines in the diagram. In general, the (a,b) box consists of all the (x,y) squares such that

$$1 \leqslant x \leqslant a \quad \text{and} \quad 1 \leqslant y \leqslant b.$$

Each of the $m\,n$ different integer pairs (x,y), $(1 \leqslant x \leqslant m,\ 1 \leqslant y \leqslant n)$ is written once on a slip of paper, which is then placed in a hat. A and B play the following game. A selects a slip of paper from the hat, notes the integer pair written on it [(c,d) say] and returns it to the hat. Thereupon, B also draws a slip; (e,f) say. If the (e,f) square lies inside the (c,d) box A pays B \$1 otherwise B pays A \$3.

Is this a fair game? If not, how much should B pay A (instead of the \$3) to make it fair?



Answer: The (e,f) square lies in the (c,d) box if and only if $e \leqslant c$ and $f \leqslant d$. Note that there are m^2 pairs of integers x_1, x_2 with $1 \leqslant x_1 \leqslant m$, $1 \leqslant x_2 \leqslant m$ but only $\tfrac{1}{2}m(m+1) = 1 + 2 + \ldots + m$ of these are such that $x_2 \leqslant x_1$. Thus, if x_1, x_2 are selected independently at random in this range the probability that $x_2 \leqslant x_1$ is $\tfrac{1}{2}(1 + m^{-1})$.

Likewise, if y_1, y_2 are selected independently at random in the range $1 \leqslant y_i \leqslant n$ the probability that $y_2 \leqslant y_1$ is $\tfrac{1}{2}(1 + n^{-1})$.

If all four selections are made independently at random the probability that both $x_2 \leqslant x_1$ and $y_2 \leqslant y_1$ is the product of these, namely

$$p = \tfrac{1}{4}(1 + m^{-1})(1 + n^{-1}) > \tfrac{1}{4}.$$

Thus the square will be in the selected box with probability $p > \tfrac{1}{4}$. So the game favours A.

B should pay A $\frac{1-p}{p} = \frac{3mn - m - n - 1}{mn + m + n + 1}$ dollars.

5. Four football teams A, B, C, D play against each other in a tournament. Each pair of teams plays exactly one match, and the winning team receives 2 points, the losing team none. If the match is a draw, both receive 1 point.

John switches on the wireless just when the announcer sums up the results: "... team D came fourth. So no two teams received the same number of points and the only draw was the game A against B".

John is disappointed because his favourite team was not even mentioned. Show that it is possible to find out the placing of team C and the number of points received by C from the given data.

Answer: Suppose A received a points, B received b points, C received c points and D received d points. We may suppose that $a > b$. Then (since only one match was a draw) a, b are odd and c, d are even.

$a + b + c + d$ = total number of points available
$= 12$ (1)

Since D came fourth, $d \leqslant 12/4 = 3$. Thus $d = 0$ or 2. If $d = 2$, the only solution of (1) is $a = 5$, $b = 3$, $c = 2$, which is impossible since no two teams received the same number of points.

Thus $d = 0$ and so A, B, C all beat D. Thus

$$6 > a > b \geqslant 3 \quad \text{and} \quad 6 \geqslant c \geqslant 2.$$

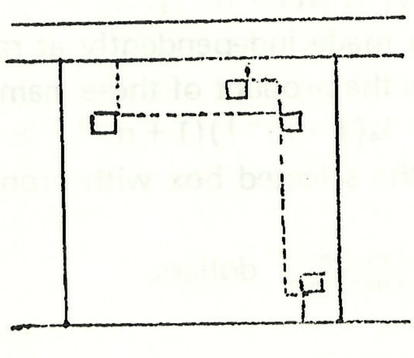
Hence $a = 5$, $b = 3$ and from (1), $c = 4$.

Thus C came second with 4 points.

6. There are two parallel highways running along the sides of a square paddock of width 1 km. There are four houses on the paddock and we want to construct footpaths running parallel to the sides of the paddock so that the occupants of the houses can walk to both highways on these footpaths.

Prove that this can always be accomplished so that the total length of the paths is not more than $2\frac{1}{2}$ km. Find an arrangement of the houses so that it is impossible to construct the required footpaths with total length less than $2\frac{1}{2}$ km.

NOTE: Pedestrians are not allowed to walk on the highways.



Answer: See Junior question 6.