Solutions to Problems 231–240 in Vol. 10 No. 1. The names of the successful problem solvers will appear in the next issue (Vol. 10 No. 3).

Junior

J231 *(submitted by J. Pike)* A man goes to an auction with \$100 and buys exactly 100 animals. The bulls sell at \$5 each, the sheep sell at \$1 each and the hens sell at 5 cents each. If he spends the whole \$100 and buys some of each of the above three animals, how many of each does he buy? *(Maybe you can send in a similar problem? – Editor)*

Answer: We require a solution in positive integers of the simultaneous equations:–

$$b + s + h = 100 \qquad (1)$$

$$5b + s + \frac{h}{20} = 100 \qquad (2)$$

in which b stands for the number of bulls, s the number of sheep, and h the number of hens. Subtracting, we obtain

$$4b - \frac{19h}{20} = 0; \text{ whence}$$

$$80b = 19h. \qquad (3)$$

It follows that b is a multiple of 19, and since, from (2), $5b < 100$, we must have $b = 19$. Now from (3), $h = 80$ and from (1), $s = 1$.

J232 In the following calculation, all but one digit is indicipherable. Replace each of the dots by the digit that should have been there.

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        • • •
      x • • •
    ---------
      • • • •
    • • • •
  • • • •
  -----------
  • • • • • •
  +   1 • • •
  -----------
• • • • • • •
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Answer: The product of the two 3 digit numbers is at least 998001, since it falls short of a 7 digit number by less than 2000. Since

$$999^2 = (1000-1)^2 = (1000)^2 - 2 \times 1000 + 1 = 998{,}001$$

we see that the calculation cannot be other than

$$\begin{array}{r} 999 \\ \underline{999} \\ 8991 \\ 8991 \\ \underline{8991} \\ 998001 \\ \underline{1999} \\ \underline{1000000} \end{array}$$

Intermediate

I233 The following are n ($\geqslant 4$) simultaneous equations in the pronumerals $x_1, x_2, x_3, \ldots x_n$.

$$\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 + x_5 = 0 \\ x_3 + x_4 + x_5 + x_6 = 0 \\ \hline \hline x_{n-3} + x_{n-2} + x_{n-1} + x_n = 0 \\ x_{n-2} + x_{n-1} + x_n + x_1 = 0 \\ x_{n-1} + x_n + x_1 + x_2 = 0 \\ x_n + x_1 + x_2 + x_3 = 0 \end{array}$$

For what values of n is the solution unique? When it is not unique, find the most general solution.

Answer: From the first two equations $x_1 = x_5$. Similarly $x_k = x_{k+4}$ if both subscripts are less than n, and (from first equation and last four equations) $x_{n-3} = x_1$, $x_{n-2} = x_2$, $x_{n-1} = x_3$ and $x_n = x_4$. If $n = 4m + 1$, these relations give

$$x_1 = x_5 = x_9 = \ldots = x_n = x_4 = x_8 = \ldots = x_{n-1} = x_3 = \ldots = x_{n-2} = x_2 = \ldots = x_{n-3}.$$

Thus all the x's are equal and the only solution is $x_i = 0$ for every i. The same result is obtained if n = 4m + 3.

If n = 4m,

$$x_1 = x_5 = x_9 = \dots = x_{n-3} = a, \text{ say.}$$

Also

$$x_2 = x_6 = \dots = x_{n-2} = b$$
$$x_3 = x_7 = \dots = x_{n-1} = c$$
$$x_4 = x_8 = \dots = x_n = d$$

Provided a + b + c + d = 0, all the equations are satisfied. (Any three of a,b,c,d may be chosen arbitrarily).

If n = 4m + 2, we have

$$x_1 = x_5 = x_9 = \dots = x_{n-1} = x_3 = x_7 = \dots = x_{n-3} = a \text{ (say)}$$
$$x_2 = x_6 = \dots = x_n = x_4 = \dots = x_{n-2} = b$$

The equations are all satisfied provided b = —a.

1234 A man views the Pentagon in Washington through binoculars from a large distance away. What is the probability that he can see three sides?

Answer: In Figure 1, an observer in the region shaded with lines can see only the two sides MN and NH of the regular pentagon. An observer in the region marked with dots can see the three sides MN, NH and HK. The angle $\angle LAK$ is easily seen to be $\pi/5$ radians. In Figure 2, a circle with centre O (the centre of symmetry of the pentagon) and of radius r has been drawn. A viewer on this circle can see two sides of the pentagon if he is situated on the arc PQ (or on one of four equal arcs), and can see three sides if he is on one of the remaining five arcs such as QR. Provided r is sufficiently large so that the dimensions of the pentagon are negligible in comparison, the angle subtended at O by PQ will be almost the same as the angle it subtends at A, viz. $\pi/5$. Similarly $\angle QOR \cong \angle QBR = \pi/5$. Hence there will be approximately equal probability that a randomly chosen point on the circle will be on one of the five arcs like PQ, or on one of the five like QR. Thus the required probability is approximately ½.

More accurately, the angle 2θ subtended at O by QR falls short of $\pi/5$ by 2ϕ, since $\theta + \phi = \pi/10$ (see Figure 3). Applying the sine rule to the triangle OBR gives

$OR \,.\, \sin\angle BRO = OB \,.\, \sin\angle OBR$

i.e. $\quad r \sin\phi = d \sin\frac{9\pi}{10}$

$$= d \sin\frac{\pi}{10}$$

Thus the probability of seeing three sides

$= 5 \times 2\theta/(2\pi)$

$= \frac{1}{2} - 5\phi/\pi$

$= \frac{1}{2} - \frac{5}{\pi}\sin^{-1}\left(\frac{d}{r}\sin\frac{\pi}{10}\right)$

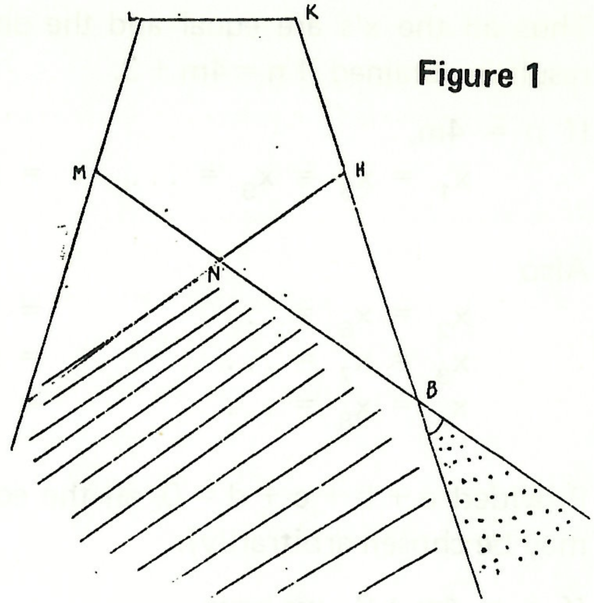


Figure 1

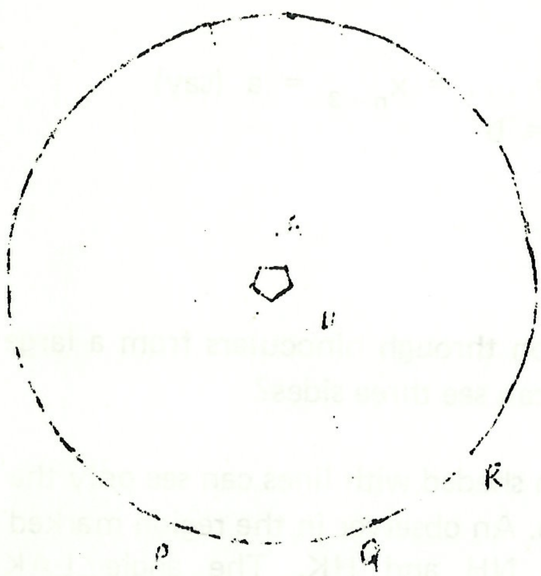


Figure 2

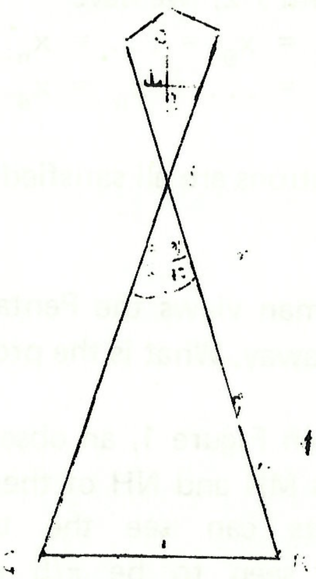
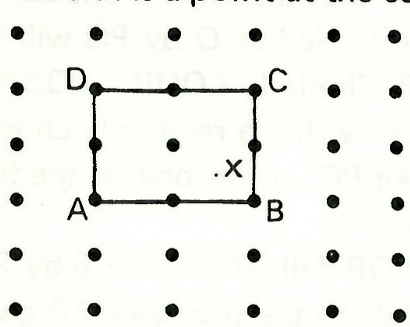


Figure 3

1235 X is a point at the centre of a square array of dots with 2n dots on each side.

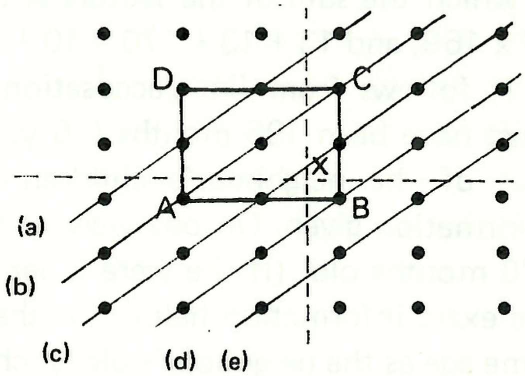


(In the diagram shown n = 3). The diagram shows a square, ABCD, whose sides lie on the rows or columns of dots, and which encloses X.

(i) How many such squares may be drawn if n = 1?, 2?, 3?, . . . k?

(ii) How many rectangles enclosing X may be drawn having sides on the rows or columns of dots?

Answer: (i) The figure shows one such square. The vertex A must be one of the k^2 points in the bottom left-hand corner of the figure, and C one of the points in the top right-hand corner. The diagonal AC of the square must lie on one of the lines (a), (b), (c), (d), (e). (In general there will be $2k-1$ such lines.) How many squares, like the one in the figure, have AC on line (b)? There are two possible points which can be chosen for A, and two for C, the choices being independent. Hence there are 2^2 such squares. A similar calculation for each of (a), (b), (c), (d), and (e) yields 1^2, 2^2, 3^2, 2^2, and 1^2 squares having AC on the respective line.



Hence, when $n = 3$ the answer required is

$$1^2 + 2^2 + 3^2 + 2^2 + 1^2.$$

If $n = k$, the number of squares is seen to be

$$1^2 + 2^2 + 3^2 + \ldots + (k-1)^2 + k^2 + (k-1)^2 + \ldots + 1^2.$$

(This answer would suffice. It is possible to simplify this expression to obtain the answer in the form $k(2k^2 + 1)/3$. Prove by mathematical induction.)

(ii) A can be chosen at any of the k^2 points in the bottom left-hand corner, and C independently at any of k^2 points in the top right-hand corner. Thus the number of rectangles is $k^2 \times k^2 = k^4$.

Senior

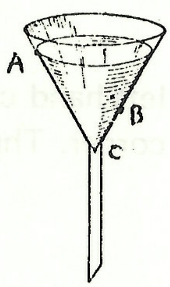
0236 When Jim Brown came home he asked his wife about the new neighbours. "Three children, including one adopted," she said. Jim wanted to know how old they were. "Work it out for yourself," she said. The product of their ages in *months* is 28,730 and the sum of their ages is exactly equal to our son's age." Jim heaved a sigh, but meekly sat down with pen and paper. After checking his calculations half a dozen times, he pointed out accusingly that he didn't have enough information. "Oh, I'm sorry I forgot to tell you that none of them is as old as our daughter." "Alright, now I know their ages," said Jim. How old are the *Brown's* children?

Answer: It is of course implicit in the problem that all ages are in a whole number of months: otherwise neither Jim Brown, nor we, could hope to find a unique solution. Of all the factorizations of 28,730 into three factors, there are only two in which the sum of the factors is the same: viz. 28,730 = 13 x 13 x 170 = 10 x 17 x 169; and 13 + 13 + 170 = 10 + 17 + 169 = 196.

It follows from Jim's accusation about lacking information that his son's age must have been 196 months (16 years 4 months). He can decide that the actual ages of the neighbour's children are 10, 17 and 169 months by the extra information given. On one view of the use of words, this means his daughter was 170 months old. (If she were older than 170 months, Jim would not have found the extra information helpful; if she were exactly 169 months, she would be the same age as the neighbour's oldest child – not older.)

It is equally possible to make a strong case that his daughter was 169 months old. In this view, if two people are the same age in years, or months, one may still be older than the other by some smaller time unit. With this use of words, Jim would have been unable to discriminate between the two sets of ages if his daughter's age were 170 months (or older of course), but could have ruled one out if his daughter's age were 169 months. Either answer is acceptable.

O237 The diagram shows a filter funnel lined with a piece of filter paper in the usual fashion, and held with its axis vertical. (The circular filter paper is folded twice, resulting in a quadrant shape. This is then opened out into a cone which fits snugly into the funnel. If you have never done any chemistry ask someone from the science class to show you.)



An ant is situated at the point A on the edge of the filter paper (whose radius OA is 10 cms) and a spot of honey is at B, on the opposite side of the funnel and 5 cms from O. The ant walks along the shortest path to the honey. How far is the ant from O when its path is horizontal? Also what is the semi-vertical angle of a conical filter funnel? (i.e. the angle between OA and the vertical axis?)

Answer: Only half the circumference of the filter paper is used to make up the circumference of the top of the cone. Hence the radius CA of this circle is half the radius OA of the filter paper. Hence $\angle$COA (the semi-vertical angle of the cone) is 30°. Also the arc AD is only a quarter of the circumference of the paper, so when the paper is opened out again, the radii OA and OD are at right angles. (See Figure

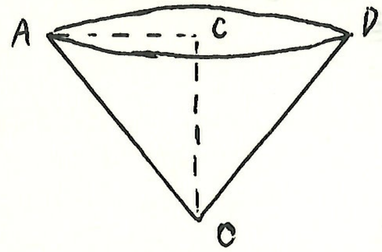


Figure 1

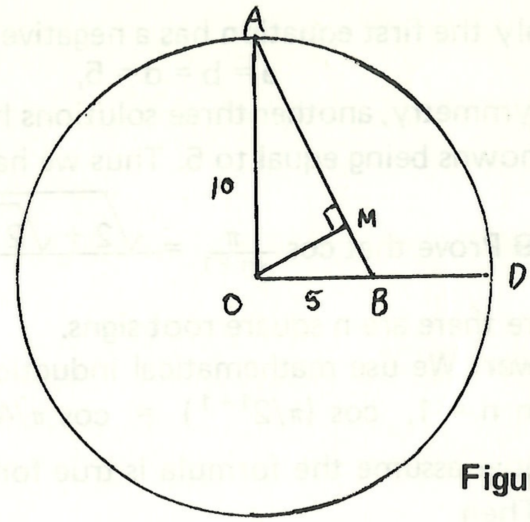


Figure 2

2.) The ant's path is the straight line from A to B. This path is horizontal when it is perpendicular to a radius of the paper, viz. at the point M. Note that OM.AB = 2 x area of $\Delta AOB = 5 \times 10$ sq. cms.

Since $AB = \sqrt{5^2 + 10^2}$ cm $= 5\sqrt{5}$ cm, $OM = 10/\sqrt{5}$ cm $= 2\sqrt{5}$ cm.

O238 Find all solutions of the simultaneous equations

$$ab + cd = 25$$
$$ac + bd = 25$$
$$ad + bc = 25$$
$$a + b + c + d = 15$$

Answer: $(a-b+c-d)^2 = (a+b+c+d)^2 - 4(ab+cd) - 4(bc+da)$

$$= 225 - 4 \times 25 - 4 \times 25$$
$$= 25.$$

Hence $a - b + c - d = \pm 5$

Similarly $a - b - c + d = \pm 5$

and $a + b - c - d = \pm 5$

Also $a + b + c + d = 15$

There are 8 ways of choosing the signs. For each choice, we have routine solution of simultaneous equations. For example if all signs are positive, we find (by adding all four equations above to get $4a = 30$)

$$a = 7\tfrac{1}{2},\ b = c = d = 2\tfrac{1}{2}.$$

By symmetry, three other solutions are $a = 2\tfrac{1}{2} = c = d$, $b = 7\tfrac{1}{2}$;
$a = b = d = 2\tfrac{1}{2}$, $c = 7\tfrac{1}{2}$;
$a = b = c = 2\tfrac{1}{2}$, $d = 7\tfrac{1}{2}$.

If only the first equation has a negative sign, we find that

$$a = b = d = 5, \qquad c = 0.$$

By symmetry, another three solutions have a, b, or d equal to zero, the other three unknowns being equal to 5. Thus we have already found all eight solutions.

O239 Prove that $\cos \frac{\pi}{2^{n+1}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}}{2}$

where there are n square root signs.

Answer: We use mathematical induction.

When n = 1, $\cos(\pi/2^{1+1}) = \cos \pi/4 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

We now assume the formula is true for n = k−1 and (for convenience) write θ for $\frac{\pi}{2^k}$. Then

$$\cos\theta = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}{2} \quad \text{with k−1 square root signs}$$

$$1 + \cos\theta = 1 + \frac{\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}{2} = \frac{2 + \sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}{2}$$

Now $\cos^2 \theta/2 = (1+\cos\theta)/2$

$$= \frac{2 + \sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}{4} \quad \text{with k−1 square root signs}$$

Hence $\cos \frac{\pi}{2^{k+1}} = \cos \frac{\theta}{2} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}}{2}$ with k square root signs.

Hence the formula is true when n = 1, and it is true for n = k whenever it is true for n = k−1. Thus it is true for all n.

O240 All entrants for a certain school chess tournament were from form IV or form V. There were nine times as many from the lower form, but between them they only scored 4 times as many points as the entrants from form V. The tournament was conducted on the Round Robin system (i.e. each entrant played all the others once, scoring 1 for a win, ½ for a draw and 0 for a loss). What was the winner's score?

Answer: Suppose there were x entrants from form V, and 9x from form IV. The 10x competitors play ½[10x(10x−1)] games altogether, and the number of points scored by the x form V entrants is 1/5 of this, viz. x(10−1). Thus the average number of wins by a form V entrant is x(10x−1)/x = (10x−1); i.e. every form V entrant must have won every game. This is impossible if there is more than one entrant from form V (they cannot both win when they meet each other). Hence x = 1 and the sole entrant from form V won with a score of 9 points.