Parabola Volume 10, Issue 3 (1974)

ZAP YOUR WAY TO HAPPINESS

Anybody who knows where he is going is happy, unhappy people wander aimlessly in circles.

A way of finding out whether a number is happy or not is simply to take each of the digits of the number, square them, then add these squares together. We can call this operation a ZAP. If we keep zapping the resultant number and this process finally ends with a result of one then we say that the number is happy.

For example, let us zap 13 and see whether it is happy or not. (We can use a special sign, $\underline{Z}$, for the operation zap!).

$$\begin{aligned} \overset{\underline{Z}}{13} &= 1^2 + 3^2 \\ &= 10. \end{aligned}$$

So we can write that $\overset{\underline{Z}}{13} = 10$.

We will now zap 10 and see what the result is.

$$\overset{\underline{Z}}{10} = 1$$

So we have:–

$$\overset{\underline{Z}}{13} = 10$$

$$\overset{\underline{Z}}{10} = 1.$$

Or writing this alternatively

$$13 \xrightarrow{Z} 10 \xrightarrow{Z} 1$$

We have our first happy number! Thirteen!

We see that 10 must also be happy. We have the set, H, of happy numbers.

$$H = \{13, 10, 1, \ldots\}$$

Are there any more?

Let's try zapping the number 7 and find if it is happy.

$$\overset{\underline{Z}}{7} = 49$$

$$\begin{aligned} \overset{\underline{Z}}{49} &= 4^2 + 9^2 \\ &= 16 + 81 \\ &= 97. \end{aligned}$$

$$\overset{\underrightarrow{Z}}{97} = 81 + 49$$
$$= 130$$

$$\overset{\underrightarrow{Z}}{130} = 1^2 + 3^2 + 0^2$$
$$= 10$$

$$\overset{\underrightarrow{Z}}{10} = 1.$$

This means that 7 is happy, and incidently all the numbers that were generated in the zapping process, since they all will finally result in 1 after a number of zaps.

So the set of known happy numbers has increased.

$$H = \{1, 7, 10, 13, 49, 97, 130, \ldots\}.$$

We see that both 13 and 130 are happy and if we think about this we come to the conclusion that placing zeroes anywhere in a happy number will also give you another happy number. Do you see why?

This means that the set, H, is an infinite set since I can have as many numbers as I like by adding zeroes to numbers such as 13.

e.g. 13, 130, 1300, 103, 1003, 1030, . . .

All of these are happy and thus elements of the set.

If we reverse the digits of a happy number we make another happy number.

For example 13 is happy and so is 31. (In fact any rearrangement of digits will preserve happiness.)

Perhaps all numbers are happy? Let us try the number 2. You may care to test this number before reading on!

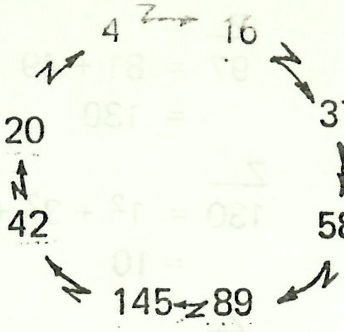
We have, using successive operations of $\underrightarrow{Z}$:–

$$\overset{\underrightarrow{Z}}{2} = 4, \quad \overset{\underrightarrow{Z}}{4} = 16, \quad \overset{\underrightarrow{Z}}{16} = 37, \quad \overset{\underrightarrow{Z}}{37} = 58, \quad \overset{\underrightarrow{Z}}{58} = 89, \quad \overset{\underrightarrow{Z}}{89} = 145, \quad \overset{\underrightarrow{Z}}{145} = 42,$$

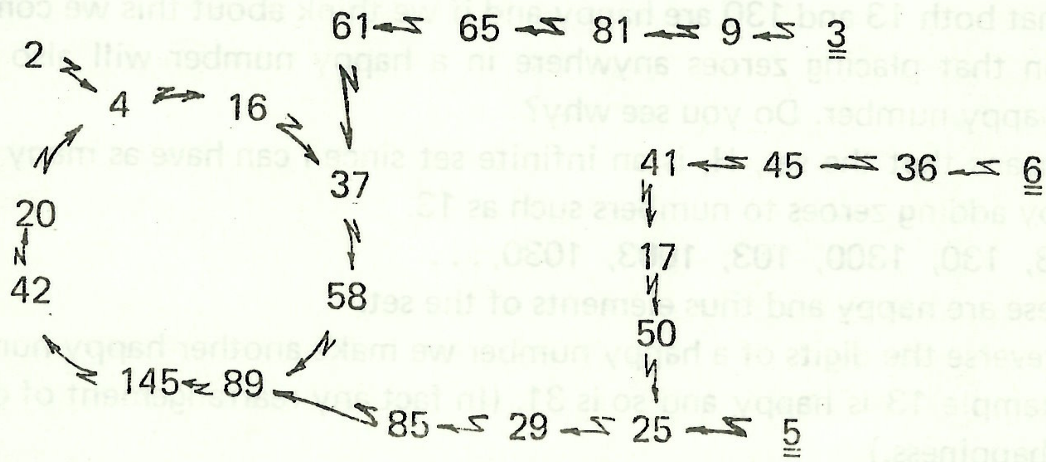
$$\overset{\underrightarrow{Z}}{42} = 20, \quad \overset{\underrightarrow{Z}}{20} = 4!!!$$

We see that 2 goes into the following cycle

$4 \xrightarrow{Z} 16 \xrightarrow{Z} 37 \xrightarrow{Z} 58 \xrightarrow{Z} 89 \xrightarrow{Z} 145 \xrightarrow{Z} 42 \xrightarrow{Z} 20 \xrightarrow{Z} 4$ with eight members in it. This cycle will never reach 1, so we have our first unhappy number and also any of the members of the cycle must be unhappy. We can represent this cycle as a network as shown overleaf.



If we further test the numbers 3, 5 and 6 we find out that these are also unhappy and, all the above can be represented by the following additions to the network.



We see that all these unhappy numbers finally end in a cycle, in fact the *same* cycle that occurred when we zapped 2.

An immediate question that presents itself is whether all unhappy numbers end up in the same eight element cycle. You can convince yourself that any very large number when zapped will reduce to a smaller number with less digits. For example the ten digit number 9999999999 becomes:

$$\overset{Z}{\overline{9999999999}} = 10 \times 81$$
$$= 810 \qquad \text{(Why?)}$$

The result is a 3 digit number. We will zap the largest 3 digit number.

$$\overset{Z}{\overline{999}} = 243$$

This is another 3 digit number. All numbers that have more than 3 digits will eventually reduce to three or less digits. To answer our question we need only to test all the numbers less than or equal to 243 to find any other cycles. This is not a hard task when you try it since a lot of numbers are eliminated because they are just happy or unhappy numbers with zeroes added or the digits reordered. If your school has a programmable calculator then, perhaps with the help of your teacher, you should be able to use the calculator for the repetitious work of testing for happy numbers. You may even be able to find the first one thousand happy numbers.

One of the most important facts about happy numbers came to light while I was testing whether a number was happy in more than one base.

Only base ten numbers have been used so far. What happens if we take the number 13_{ten} (13 base ten) and convert it to base 5.

$$13_{ten} = 23_{five}.$$

Now

$$\underline{Z}\ 23_{five} = 2^2_{five} + 3^2_{five}$$
$$= 4_{five} + 14_{five}$$
$$= 23_{five}.$$

Fantastic!?!!

13 is happy in base ten but unhappy in base 5!! Not only is it unhappy in base 5 but when zapped gives itself as a result. It also must be its *own cycle* for unhappiness in base 5. All of these are remarkable and wonderful results.

HAPPINESS IS BASE DEPENDENT

This means that happiness depends on the *numeration* system unlike the properties of primeness or evenness since these are properties of the *number* itself and not the *numeral.*

A remarkable and very important difference.

A whole new area is opened up; happiness in other bases, self-generating numbers like 23_{five} (are there any others in base five? base 6? base 7? etc?), zapping to higher powers (e.g. cubed, fourth power, etc).

I would be very happy (no pun intended) to receive any results or patterns that you may find as we believe very little is known about happiness in general.

Have a happy time!

W. Moore,
M.C.A.E.

Questions:

Can you find any consecutive happy numbers? These are called lovers.

Do you know of any number that is happy in any base?

Find a base that makes all numbers happy.

Link some of the happy numbers together in a network. (Note: this is called a tree. Why?)