A NOTE ON UNIT FRACTIONS

This note reports on some work which arose out of a discussion* of unit or Egyptian fractions. Roughly and in contemporary terms the Egyptians worked entirely with fractions with numerator 1. They were interested in expressing arbitrary fractions as sums of distinct unit fractions. For example 2/5 = 1/3 + 1/15. During the discussion it was established that for every positive integer n

$$2(2n+1) = 1/(n+1) + 1/(n+1)(2n+1)$$
$$3/2n = 1/n + 1/(2n)$$
$$3/(2n+1) = 1/(n+1) + 1/(2n+1) + 1/(n+1)(2n+1).$$

The question was asked whether fractions of the form 3(2n + 1) could be written as the sum of less than three unit fractions. For example 3/5 = 1/2 + 1/10.

After a certain amount of consideration of cases it seemed reasonable to guess 3/n cannot be written as the sum of less than three unit fractions if and only if n and each of its factors can be written in the form 3k + 1.

This can be proved as follows. Note first that

$$3/(3k+2) = 1/(k+1) + 1/(k+1)(3k+2).$$

If n = m(3k + 2), then

$$3/n = 1/m(k+1) + 1/m(k+1)(3k+2).$$

So all that needs to be proved is that 3/n, where n and each of its factors can be expressed in the form 3k + 1, cannot be expressed as 1/a + 1/b. (Obviously 3/(3k + 1) can never equal 1/a.) Assume 3/n = 1/a + 1/b. Then 3ab = n(a + b). Take a′ as the largest factor of a which is prime to n, and $n_1 = a/a'$. Similarly $b = b'n_2$ where b′ is the largest factor of b which is prime to n. Hence

$$3a'b'n_1n_2 = na'n_1 + nb'n_2.$$

Now a′ divides $3a'b'n_1n_2$ and $na'n_1$, so a′ divides $nb'n_2$. Since a′ is prime to all the factors of n, we have a′ divides b′. Similarly b′ divides a′. Thus a′ = b′.

Therefore $3a'a'n_1n_2 = na'n_1 + na'n_2$,

so $3a'n_1n_2 = nn_1 + nn_2$.

Therefore 3 divides $nn_1 + nn_2$.

But all the factors of n are in the form 3k + 1. Therefore nn_1 and nn_2 are also in the form 3k + 1 and so $nn_1 + nn_2$ is in the form 3k + 1 + 3k′ + 1 or 3(k + k′) + 2. Therefore 3 does not divide $nn_1 + nn_2$. Thus we have a contradiction and the original assumption that 3/n (where all the factors of n are in the form 3k + 1) can be expressed as 1/a + 1/b is false.

It is not too dificult to see that every fraction 4/n can be written as a sum of at most four unit fractions. Which, if any, need four unit fractions?

Mark Durie

* This discussion took place at one of the Friday evenings sponsored by the Canberra Mathematical Association and the Australian National University.

[This is a very good article. Some of our readers might like to take up Mark's challenge to find the number of unit fractions needed for 4/n — Ed.]

☆ ☆ ☆

Odds and Evens

Ask a friend to spread a handful of coins out on a table. You glance at the coins, turn your back, then ask him to simultaneously turn two coins at a time. Stress that he must turn two coins at a time and ask him to do it as quietly as possible so that you won't know how many times he turned coins.

When he has finished have him cover one of the coins with his hand. You turn around, casually glance at the uncovered coins, then tell him whether he has covered a head or a tail!

How's it done? It's quite simple really. When he first spreads out the coins you note whether there is an even or odd number of heads visible. Suppose an odd number of heads was visible. Providing he turns two coins at a time there will always be an odd number of heads visible. So when you look at the coins the second time you note whether an even or odd number of heads is visible. In this case it was odd to begin with, so if an odd number of heads is visible then he has covered a tail. If it is even, then he has covered a head.

As a variation have him turn over three coins, then two coins, then one. 3 + 2 + 1 = 6 (an even number of turns) and the method is unchanged.

W.J. Ryan

☆ ☆ ☆

Mathematical Pen-friends

Since our suggestion in the previous issue, that our readers might like to correspond with one another, we have received only one name. Obviously he does not want to write to himself, so if you are interested in writing to him, let us have your name and address.