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A LARGE CAN OF PAINT

A talk given to high school Level 1 students on June 20, 1974 by Dr J. Mack of Sydney University.

It is a matter of observation that paint tins are depressingly similar – they are all right circular cylinders of finite height. If a paint tin has height h and diameter D, then the tin when full holds an amount V of paint given by $V = (\pi D^2 h)/4$. The total area S of metal sheeting needed to make the tin is of course given by $S = \pi Dh + (\pi D^2)/2$.

A paint manufacturer usually wishes to sell paint that people will buy, which means that his product must be good enough and cheap enough to be competitive. To maximise his profit, he must certainly try to minimise his costs of labour and materials, subject to the constraints that people will work for him and that his paint will continue to sell. One such manufacturer, x say, was fairly pleased with his position, and felt certain that he had reduced his costs to a minimum. However, he employed a girl y, who quickly realised that she could reduce these costs still further. For x had left school post-arithmetic but pre-calculus, while y knew her calculus, and had observed that x's tins were usually of a shape where h was one-and-a-half times D. She offered to reduce x's costs, and x offered her in return a percentage increase in pay equal to his percentage reduction in costs. Now for a given volume V, we have

$$S = 4V/D + \pi D^2/2,$$

so that

$$\frac{dS}{dD} = -4V/D^2 + \pi D,$$

which vanishes when $D^3 = 4V/\pi$ i.e., when D = h. y accordingly advised x to switch to the new shape given by h = D. Each of x and y then worked out the percentage saving on area of metal sheeting. I leave you to check that it *is* a saving, and to find its amount.

This financial success gave y added incentive, and she began to explore other ways of applying mathematics to the paint industry. Among other ideas, she guessed that for a given volume V, a container has minimum surface area if it is ball-shaped. (Certainly, it is easy to check that a ball of volume V has less surface area than a cylinder of volume V. The general result is much tougher.) However, she realised that the public would not want ball-shaped cans of paint – they would be awkward to stir or to stand up – and they might be more expensive to make, too.

Well, after a few bright but frustratingly impractical ideas, y decided that she would let her imagination run free, and design some aesthetic but not-necessarily-practical paint cans. She did require that her containers have finite volumes V and finite surface areas S, but apart from that, she wandered into the world of mathematical shapes and curves. Not knowing about double integrals and the like, y's imagination was of course restricted to those shapes for which she could calculate V and S. She remembered learning something about solids of revolution and their volumes, and started with that. After all, there are lots of nice tapered cones – very pretty to look at either way up – which are just solids of revolution. They can be constructed as follows. Take a nice smooth positive decreasing function f, defined say for $a \leqslant x \leqslant b$. (By 'smooth', I mean a function f such that f and f' are both continuous.) Regard the Oxy-plane as part of a three-dimensional coordinate system, and rotate the curve $y = f(x)$ about the x-axis. We obtain a 'surface of revolution' which is the curved part of the surface of a 'solid of revolution'. Tilting the solid upright, we have a nice paint container.

y recalled the usual first-order approximation argument employed to find the volume of a solid of revolution. Each cross-section perpendicular to the x-axis is a disc, so that a thin slice of the solid is approximately a short right circular cylinder. If the thin slice has its base passing through the point P(x) on Ox, and has height δx along Ox, then its volume δV is approximately,

$$\begin{aligned}\delta V &\cong \text{area of base times height}\\ &= \pi (f(x))^2\, \delta x.\end{aligned}$$

Chopping the solid into a large number of such thin slices, we find

$$V \cong \Sigma \pi (f(x))^2\, \delta x$$

and, by virtue of the definition of a definite integral, V is given by

$$V = \int_a^b \pi (f(x))^2\, dx.$$

To calculate this, we need an explicit function f. A simple choice for f (in fact for a whole family of functions f) is to take $f(x) = x^{a}$, for a fixed a less than 0. The corresponding f is then smooth and decreasing. Choosing $a = 1$ and $b = X$, we find the volume V is

$$V = \int_1^X \pi x^{2a}\,dx.$$

This volume clearly depends on X and on a; so we shall denote it by $V(X,a)$. We can find it explicitly:

$$V(X,a) = \frac{\pi}{2a+1}(X^{2a+1}-1) \qquad \text{if } a \neq -\tfrac{1}{2}.$$

$$V(X,-\tfrac{1}{2}) = \pi \ln X.$$

y noted that if $a_1 < a_2 < 0$, then $\pi x^{2a_1} < \pi x^{2a_2}$ for $x > 1$, so that $V(X,a_1) < V(X,a_2)$ for each $X > 1$. Since $V(X,-\tfrac{1}{2}) = \ln X$ is unbounded as X tends to infinity, it follows that for $-\tfrac{1}{2} \leqslant a < 0$, the corresponding solids of revolution have arbitrarily large volumes as their heights increase. Now y thought these containers beautiful only when they were 'long and thin'. Since there would then not be enough paint in the world to fill any beautiful container obtained from a curve $y = x^a$ with $a \geqslant -\tfrac{1}{2}$, she reluctantly discarded them.

For $a < -\tfrac{1}{2}$, she found immediately that

$$V(X,a) = \frac{\pi}{|2a+1|}(1 - X^{-|2a+1|}) < \frac{\pi}{|2a+1|}$$

for every $X > 1$. Such containers therefore have, for each a, a bounded volume, no matter how high they are, and are clearly possible ideal paint cans. What of their surface areas? y did not recall any formula, so she set about constructing one, by analogy with the volume formula.

At first she reasoned that the area of the curved surface of a thin slice would be approximately that of a thin circular cylinder. This gave the formulae

$$\delta S \cong 2\pi f(x)\,\delta x,$$
$$S \cong \Sigma\, 2\pi f(x)\,\delta x,$$
$$S = \int_a^b 2\pi f(x)\,dx.$$

Being a practical girl, she checked this against the surface area of the (ordinary) cone given by $f(x) = 1-x$ for $1 \leqslant x \leqslant 2$, and got too small an answer. She then drew a more careful picture of a thin slice, and realised that, to the first order, its curved surface area is that of cylinder of radius $f(x)$ and of height the slant height $\sqrt{((\delta x)^2 + (\delta y)^2)}$. This gave

$$\delta S \cong 2\pi f(x) \sqrt{((\delta x)^2 + (\delta y)^2)}$$
$$= 2\pi f(x) \sqrt{(1 + (\delta y/\delta x)^2)}\, \delta x,$$

leading in the limit to the formula

$$S = \int_a^b 2\pi f(x) \sqrt{(1 + (f'(x))^2)}\, dx,$$

which certainly works for a simple cone.

Using the family $y = x^a$ $(a < -\frac{1}{2})$, we find that

$$S(X,a) = \int_1^X 2\pi x^a \sqrt{(1 + a^2 x^{2a-2})}\, dx$$

gives the curved surface area of the solid with volume $V(X,a)$. This integral is obviously nasty, but y observed that she could easily find an estimate for it. For, since the expression under the radical sign is clearly greater than 1 for $x \geqslant 1$ (and any a), it is clear that

$$S(X,a) > \int_1^X 2\pi x^a 1\, dx.$$

y now discovered the following curious fact. Pick an a such that $-1 \leqslant a < -\frac{1}{2}$, say $a = -2/3$. Then

$$V(X,-2/3) < \frac{\pi}{|-4/3 + 1|} = 3\pi$$

for all $X > 1$, while

$$S(X,a) > \int_1^X 2\pi x^{-2/3}\, dx = 6\pi(X^{1/3} - 1)$$

for all $X > 1$. Thus while $V(X,-2/3)$ remains *bounded* as X tends to infinity, $S(X,-2/3)$ *increases without bound!*

Returning from the world of mathematics to the world of paint, y discovered the following curious consequence of this fact. Fill the container corresponding to $V(X,-2/3)$ with best quality paint. Now even best quality paint must be applied with a minimum positive thickness, Δ say, in order to coat a surface. So the maximum surface area S^* that a volume $V(X,-2/3)$ of best quality paint can cover is

$$S^* = V(X,-2/3)/\Delta.$$

If X is chosen so that $S(X,-2/3) > S^*$, then there is insufficient best quality paint in the container to paint it! This can be achieved by choosing X so that

$$6\pi(X^{1/3}-1) > \frac{3\pi}{\Delta},$$

i.e., so that $$X > (1 + 1/(2\Delta))^3.$$

Choose $X = X_0 = 2(1 + 1/(2\Delta))^3$. The container given by the curve $y = x^{-2/3}$, $1 \leqslant x \leqslant X_0$, does not hold enough paint to paint itself. Truly, it is a large can of paint.

y noticed and solved the following paradox. (Can you?) The paint fills our large can, and so covers its inside surface. Being a mathematical can, its inside and outside surfaces have the same area. Why can't the paint coat the outside surface?

Doodling

Did you read the article on primes in the first issue of Parabola this year? One bright young fellow who read it was most interested in the statement that all primes greater than 5 are either 1 less than a multiple of 6, or 1 more than a multiple of 6. He took out a pencil and paper and began extending the list of twin-primes given in the article . . . (29,31), (41,43), (59,61), (71,73). He then realised that if a pair of primes are to be "twins", the first must be 1 less than a multiple of 6 and the second 1 more than the multiple of 6. "Aha," he thought, "if I add twin-primes, the sum must be a multiple of 12."

Feeling pretty pleased with himself he took another look at the list of primes given in the article. He saw that if he took a prime that was one less than a multiple of 6 and another that was one more than a multiple of 6, then their sum would be divisible by 6. He then found an easy method of finding two primes whose sum is divisible by 4.

Being a bit more persistent than usual he searched a little longer and came up with an easy method of finding two primes whose sum is divisible by 30. Can you?

W.J. Ryan

Proving 10 an even number

9 = ix

6 = six

3 = –s

7 = seven

∴ 10 = even

Attributed to
Augustus de Morgan
(1806 – 1871)