I255 (i) Find a 6-digit number which is multiplied by the factor 6 if the final 3 digits are removed and placed (without changing their order) at the beginning.

(ii) Show that there is no 8 digit number which is increased by the factor 6 if the final 4 digits are transferred to the beginning.

Open

O256 Simplify the following expression

$$(1+\tfrac{1}{3})(1+\tfrac{1}{9})(1+\tfrac{1}{81})(1+\tfrac{1}{3^8})\ \ldots\ (1+\tfrac{1}{3^{2^n}}).$$

O257 The equation $x^5 + y^2 = z^4$ has $x = 2$, $y = 7$, $z = 3$ as a solution in integers. Are there any other solutions in integers?

O258 Show that if p is an odd prime, it divides the difference

$$[(\sqrt{5}+2)^p] - 2^{p+1}.$$

Note that [x] means the integer n such that $n \leqslant x < n + 1$.

O259 *(Submitted by M. Durie)* You have a beam balance, but no weights, and a collection of 12 similar coins, one of which however is counterfeit. It is a different weight from the good coins, but you do not know whether it is heavier or lighter. Locate the bad coin and its relative weight in three weighings.

O260 If n and k are any positive integers, show that

$$\frac{1}{n} - {}^kC_1\frac{1}{n+1} + {}^kC_2\frac{1}{n+2} + \ldots + (-1)^j\,{}^kC_j\frac{1}{n+j} + \ldots + (-1)^k\frac{1}{n+k}$$

is equal to $\frac{1}{l.c.m.\{D_0, D_1, \ldots, D_k\}}$ where D_j is the denominator when $\frac{{}^kC_j}{n+j}$ is put in lowest terms, (j = 0, 1, 2, . . ., k). $l.c.m.\{D_0, D_1, \ldots, D_k\}$ means the smallest number which $D_0, D_1, \ldots, D_k$ will all divide.

Solutions to Problems 241–250 in Vol. 10 No. 2.

Junior

J241 By inserting brackets in

$$1 \div 2 \div 3 \div 4 \div 5 \div 6 \div 7 \div 8 \div 9,$$

the value of the expression can be made to equal 7/10. How?
Also find the largest value and the smallest value that can be obtained by insertion of brackets.

Answer: First of all note that, as the expression starts off $1 \div 2$, the 2 must end up in the denominator of the answer. To get a value of 7/10, we must put the 7 in the numerator and the 5 in the denominator.

(a) If the 8 is placed in the numerator, it must be cancelled by the even digits 4 and 6 in the denominator. The factor 3 in the 6 will then need to be cancelled by putting the 9 in the numerator and the 3 in the denominator.

i.e. $\frac{1.7.8.9}{2.3.4.5.6} = ((((1 \div 2) \div 3) \div 4) \div 5) \div (((6 \div 7) \div 8) \div 9).$

(b) If the 8 is placed in the denominator, all the digits in (a) will have to be around the other way.

i.e. $\frac{1.3.4.6.7}{2.5.8.9} = 1 \div (((2 \div 3) \div (4 \div (5 \div 6))) \div ((7 \div 8) \div 9)).$

The largest value is obtained by placing as many digits as possible in the numerator (remember that 2 must go in the denominator!)

i.e. $\frac{1.3.4.5.6.7.8.9}{2} = 1 \div (((((((2 \div 3) \div 4) \div 5) \div 6) \div 7) \div 8) \div 9).$

Similarly, the smallest value is

$$\frac{1}{2.3.4.5.6.7.8.9} = (((((((1 \div 2) \div 3) \div 4) \div 5) \div 6) \div 7) \div 8) \div 9.$$

J242 Prove that the sum of two consecutive positive integers and the sum of their squares are relatively prime (i.e. have no common factor except 1).

Answer: Let the consecutive integers be n and n + 1. We are asked to show that $[n^2 + (n+1)^2] = 2n^2 + 2n + 1$ and $[n + (n+1)] = 2n + 1$ are relatively prime. If d divides both these numbers, it clearly divides $2n + 1$ and

$$2n^2 + 2n + 1 - n(2n+1) = n + 1.$$

Since d divides both $n + 1$ and $2n + 1$, it divides

$$2(n+1) - (2n+1) = 1$$

and the result follows.

Intermediate

I243 Prove that if $a(y + z) = x$, $b(z + x) = y$, $c(x + y) = z$ and at least one of the numbers x,y,z is not equal to zero, then

$$ab + bc + ca + 2abc - 1 = 0.$$

Answer: The equations may be rewritten as

$$-x + ay + az = 0 \qquad (1)$$
$$bx - y + bz = 0 \qquad (2)$$
$$cx + cy - z = 0 \qquad (3)$$

Adding b times (1) to (2) and c times (1) to (3) eliminates x obtaining

$$(ab-1)y + (b + ab)z = 0 \qquad (4)$$
$$(c + ac)y + (ac-1)z = 0 \qquad (5)$$

Eliminating y between (4) and (5) gives

$$[(c + ac)(b + ab) - (ab-1)(ac-1)]\,z = 0$$

which simplifies to

$$[ab + bc + ca + 2abc - 1]\,z = 0 \qquad (6)$$

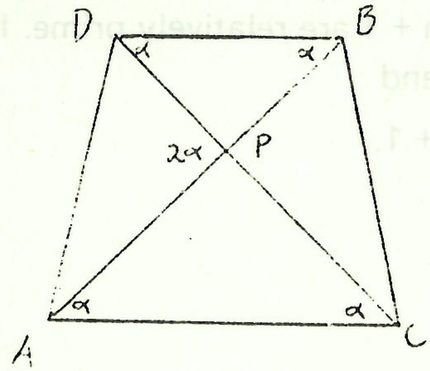
Similarly, elimination of z between (4) and (5) yields

$$[ab + bc + ca + 2abc - 1]\,y = 0 \qquad (7)$$

If $ab + bc + ca + 2abc - 1 \neq 0$ we obtain $z = y = 0$ from (6) and (7), and then $x = 0$ from (6) and (7), and then $x = 0$ from (1), contradicting the data. Hence the result.

1244 A, B, C, D are points in a plane, no three of them being collinear, with AB = CD and AD = BC. Prove that, if the segments AB and CD intersect, then the segments AC and BD are parallel.

Answer: Let AB and CD intersect at P. ΔACD is congruent to ΔCAB, since the side AC is common, while AD* = CB* and CD* = AB* (given). Hence $\angle ACD = \angle CAB$ and each is half their sum, the exterior opposite angle $\angle APD$ of the ΔPAC.



Similarly $\Delta ABD \equiv \Delta CBD$, yielding $\angle ABD = \angle CDB = \frac{1}{2}\angle APD$.

It follows that $\angle DBA = \angle BAC$, and as these are alternate angles made by the transversal AB with BD and AC, these lines are parallel.

1245 Let p and q be successive odd prime numbers (i.e., if $p < k < q$, then k is composite). Show that $p + q$ has at least three (not necessarily distinct) prime factors. Further if $p \neq 3$ and $q = p + 2$ (i.e., p and q are twin primes) show that 6 is a factor of $p + q$.

Answer: (i) Since p and q are both odd p+q is even, and $n = \frac{1}{2}(p + q)$ is an integer. Since $p = \frac{1}{2}(p + p) < \frac{1}{2}(p + q) < \frac{1}{2}(q + q) = q$, n lies between successive primes p and q, so must be composite, i.e. has at least 2 prime factors.

Hence $p + q = 2n$ has at least 3 prime factors.

(ii) Let the remainder when p is divided by 3 be r. ($r = 0, 1,$ or 2). Now $r \neq 0$ since p is a prime other than 3 and therefore is not exactly divisible by 3. If $r = 1$ (i.e. $p = 3k+1$) then $q = p + 2 = 3k + 3 = 3(k + 1)$.
But q is not divisible by 3 and so $r \neq 1$.

We must have $r = 2$ and $p = 3k + 2$ for some integer k. Then $q = p + 2 = 3k + 4$ and $p + q = 6k + 6 = 6(k + 1)$.

Open

O246 Let p be an odd prime, let $p_1 = p + 2$, $p_2 = p_1 + 4, \ldots, p_n = p_{n-1} + 2n, \ldots$, the sequence continuing until a composite number is reached, e.g. $p = 5$, $p_1 = 7$, $p_2 = 11$, $p_3 = 17$, $p_4 = 25$. Is the sequence so generated always finite? If so, what is its maximum length, in terms of p?

Answer: $p_n = p + (2 + 4 + 6 + \ldots + 2n) = p + (n + 1)n$ [where we have used a well known formula for summing an Arithmetic Progression].

It is clear that when $n = p-1$, $p_n = p + p(p-1) = p^2$.

Thus the sequence obtained is always finite. It cannot contain more than $p-1$ primes before a composite number is reached.

O247 Of three cards, one is green on both faces, one white on both faces, whilst the third is green on one side and white on the other. They are placed in a hat, one is withdrawn and placed on a table. If the visible face is green, what is the probability that the other face is also green?

Answer: There are 3 green faces altogether on the cards, and there is equal probability that the exposed face is each of these. Two of the green faces are on the one card and when turned over are still green. Only one of them has a white face on the other side. Hence the probability that the other face is green is 2/3.

O248 Let P be any property defined on the set of positive integers $N = \{1,2,3,4, \ldots\}$. (For example, P could be the property ". . is even", or ". . . is odd", or ". . . is a prime number").

Let p_k denote the k'th positive integer (in order of magnitude) which has property P, and let $\pi(k)$ denote the number of natural numbers less than k having property P. (For example, if P is the property ". . . is even",

$$p_4 = 8 \text{ and } \pi(11) = \pi(12) = 5.)$$

Show that any positive integer r may be expressed either in the form $\pi(n) + n$ or else in the form $p_n + n$ for some positive integer n.

Answer: We prove the result by induction: i.e. we show (i) that 1 is so expressible; and (ii) that if k is so expressible, so is k+1.

(i) $1 = 1 + \pi(1)$ since $\pi(1)$ is always 0, whatever property P might be, as there are no positive integers less than 1.

(ii) Now suppose that k is expressible either by

$$k = \pi(n) + n \text{ or by } k = p_m + m.$$

We are finished if we show that k + 1 is also so expressible.

Case 1. $k = \pi(n) + n$, and n does not possess property P.
Then $\pi(n+1) = \pi(n)$ and

$$k + 1 = \pi(n) + n + 1 = \pi(n+1) + (n+1).$$

Case 2. $k = \pi(n) + n$ and n possesses property P.
Then $n = p_m$ where $\pi(n) = \pi(p_m) = m-1$.
Hence $k + 1 = (m-1) + p_m + 1 = p_m + m$.

Case 3. $k = p_m + m$.
Then $k + 1 = p_m + 1 + m$
$= n + \pi(n)$ where $n = p_m + 1$.

O249 Let $P(x) \equiv a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a polynomial with coefficients a_i which are all integers. If $P(x) = 0$ for 4 different integer values of x, show that the value of P(x) is never a prime number for any integer value of x.
Construct a polynomial with integer coefficients which vanishes for 3 distinct integer values of x and which has the value 11 at a fourth integer value.

Answer: (i) Let a be an integer such that $P(a) = 0$. By the factor theorem $P(x) = (x-a)P_1(x)$ where $P_1(x)$ is a polynomial of degree (n−1) whose coefficients are still integers.

If b is a second integer such that $0 = P(b) = (b-a)P_1(b)$ we see that $P_1(b) = 0$ and the argument can be repeated. Eventually using the 4 distinct integers a, b, c, d which are zeros of P(x) we obtain where $P_4(x)$ is a polynomial of degree n−4 with integer coefficients.

If any integer is substituted for x, (1) expresses P(x) as the product of 5 integers the first 4 of which are different from each other. It is possible for 2 of these to be +1 and −1 and for $P_4(x)$ to be ±1 but there are still 2 integer factors which cannot be equal to ±1. Hence P(x) cannot be prime.

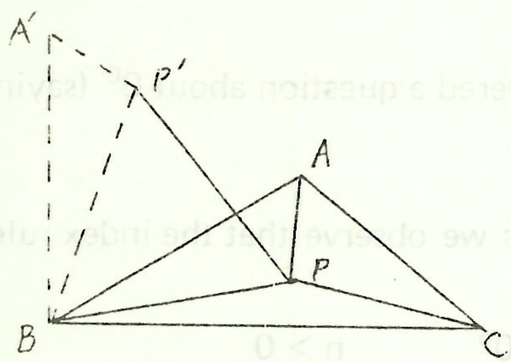
(ii) $Q(x) = (x-1)(x+1)(x-11)$ is such a polynomial.

$$Q(1) = Q(-1) = Q(11) = 0 \text{ and } Q(0) = 11.$$

O250 P is a point inside an acute angled triangle ABC. Where must P be so that the sum of the three distances PA, PB, and PC is least?

Answer: The following solution was discovered by J.E. Hoffmann in 1929 (over 300 years after the problem was invented!) and can be found in the book "Introduction to Geometry" by H. Coxeter.

Let P be any point inside the triangle and rotate the triangle APB through an angle of 60° about the point B. This will give us another triangle A'P'B and ΔA'P'B ≡ ΔAPB. Thus AP* = A'P'* . . . (1)



Since BP'* = BP*, ∠BP'P = ∠BPP'.

But ∠PBP' = 60° and the angles of a triangle sum to 180°. Thus ∠BP'P = ∠BPP' = 60° and so ΔBPP' is equilateral. So

$$BP^* = PP'^* \quad \ldots (2)$$

From (1) and (2), we get

$$PA^* + PB^* + PC^* = A'P'^* + P'P^* + PC^* \quad \ldots (3)$$

Since the shortest distance from A' to C is a straight line, the expression in (3) is minimal when ∠A'P'P = ∠P'PC = 180°. Since ∠BP'P = ∠P'PB = 60°, we must have ∠APB = ∠A'P'B = 180°−60° = 120° and ∠BPC = 180°−60° = 120°.

Thus the sum of the distances PA, PB, PC is least when the point P is placed so that ∠APC = ∠CPB = ∠BPA = 120°.

Solvers of Problems 231–240

Due to the fact that Vol. 10 No. 1 was published so late, the names of the solvers of the problems in that issue were not printed in Vol. 10 No. 2. They appear below:

V. Arvanitis (Smith's Hill Girls' High) 238.

M. Betts (Marist Brothers', North Sydney) 236.

P. Burge (Marist Brothers', North Sydney) 231, 232.

F. Carnovale (Marist Brothers', North Sydney) 238, 239, 240.

S. Cogger (East Hills Boys' High) 238.

M. Durie (No school given) 236, 238, 239, 240.

A. Fekete (Sydney Grammar) 233, 235.

G. Middleton (Marist Brothers', North Sydney) 236, 238.

D. Powers (Fort Street Boys' High) 236, 237, 238, 239.

L. Wootton-McDonald (James Ruse Ag. High) 231.

☆ ☆ ☆

What is 0^0?

At the recent 1st level day I incorrectly answered a question about 0^0 (saying it was undefined). My apologies to the answeree.

It is appropriate to define $0^0 = 1$. To see this we observe that the index rules require:

$$0^0.0^n = 0^{0+n} = 0^n \qquad n > 0$$

so $1.0 = 0$ is correct;

and $$(0^0)^m = 0^{0m} = 0^0$$

so $1^m = 1$ is correct.

(The last equation shows that $0^0 = 0$ is *not* a possible definition because we require $(0^0)^m = 0^0$ also for $m < 0$.)

One might well ask why, really, $0^0 = 1$. The reason is that, technically, one defines

$$a^\beta = e^{\beta \log a}.$$

Then by 0^0 one should mean $0^0 = \lim_{x \to 0} e^{x \log x}$.

Because $\lim_{x \to 0} x \log x = 0$, it follows that $0^0 = e^0 = 1$.

Dr A.J. van der Poorten