Parabola Volume 16, Issue 1 (1980)

THE ETERNAL TRIANGLE

When I think of Euclid even now
I have to wipe my sweaty brow.
— C.M. Bellman.

The Problem

Let ABC be a triangle with sides of length a, b and c, as shown in Figure 1. Let α, β and γ be real numbers and measure off the distances α, β and γ respectively from A, B and C along the sides AB, BC and CA of the triangle to give the points S, V and Y shown in Figure 2. Similarly, measure off the distances $-\alpha$, $-\beta$ and $-\gamma$ from A, B and C along the sides AC, BA and CB to get the points T, W and Z. (Note that a positive distance along AB, say, is measured in the direction from A to B and a negative distance is measured in the opposite direction from B to A. Figure 2 shows the case with $\alpha < 0$ and $\beta, \gamma > 0$.) Our problem is to decide when the three lines ST, VW and YZ are concurrent.

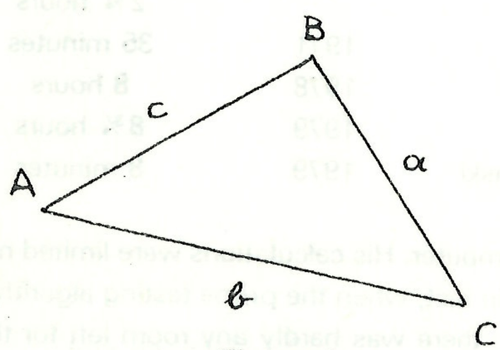

Figure 1

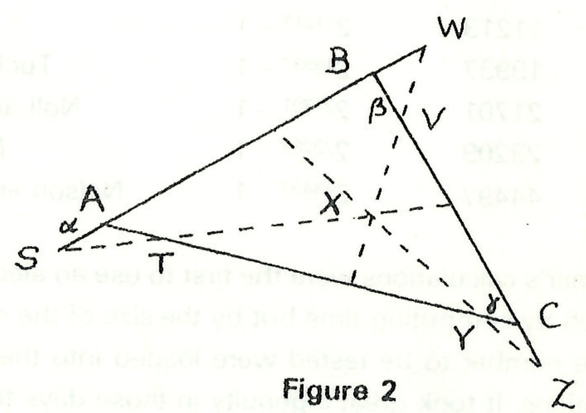

Figure 2

Background to the problem

Our problem was suggested by a letter from David McGrath (The King's School, Parramatta) in which he writes as follows.

Take a scalene triangle ABC with sides a, b and c, as in Figure 1. From the point C, mark off segments CC′ and CC″ along CA and CB with length c. (See Figure 3.) Similarly, mark off segments BB′ and BB″ along BC and BA with length b and segments AA′ and AA″ along AB and AC with length a. Finally, mark in the following midpoints: P is the midpoint of BC, Q is the midpoint of AC, R is the midpoint of AB, S is the midpoint of AB″, T is the midpoint of AC′, V is the midpoint of BC″, W is the midpoint of BA′, Y is the midpoint of CA″ and Z is the midpoint of CB′.

We want to show that the points S, T and P are collinear, the points W, V and Q are collinear, the points Z, Y and R are collinear and, most interestingly, the lines SP, WQ and ZR are concurrent. This was the problem posed in Parabola, Volume 15, Number 1. (See "Triangulation" by D. McGrath.) Here is David's solution; an abridged and somewhat altered version was given in our last issue.

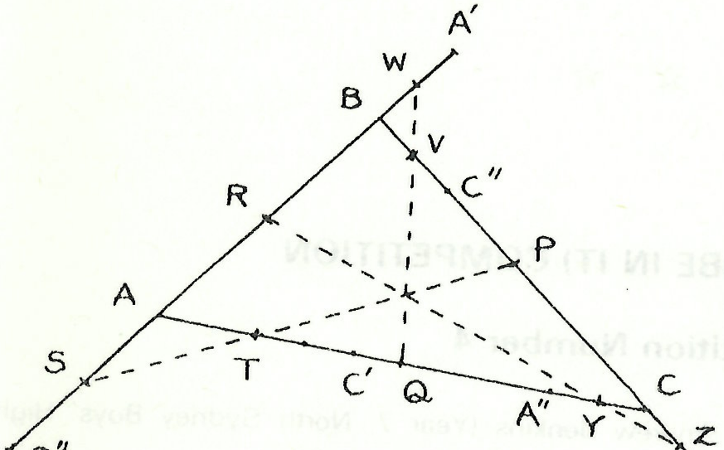

Figure 3

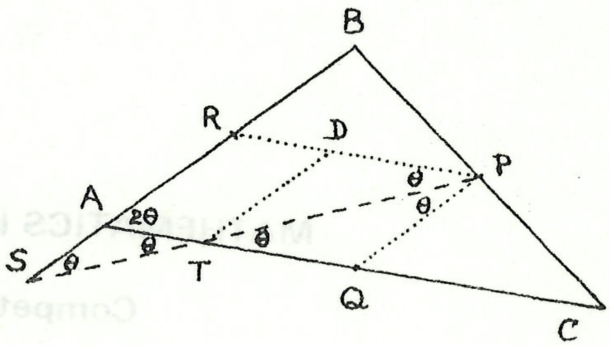

Figure 4

Consider first the collinearity of the points S, T and P. From the definition of the points S and T,

$$AS = \tfrac{1}{2}(b-c), \quad AT = \tfrac{1}{2}(b-c), \tag{1}$$

so the triangle AST is isosceles. Set angle AST = angle ATS = θ, so that angle BAC = 2θ. (See Figure 4.) Since ARPQ is a parallelogram, we also have angle RPQ = 2θ. Now, from (1),

$$AR = \tfrac{1}{2}c = \tfrac{1}{2}b - AT = AQ - AT = TQ,$$

so, by drawing TD parallel to AB, we form a rhombus TDPQ. This means that the diagonal TP bisects the angles at T and P and, since angle DTQ = angle RPQ = 2θ, we get angle PTQ = θ. But now we have angle ATS = angle PTQ, so STP must be a straight line. In the same way, we can show that the points W, V and Q are collinear and that the points Z, Y and R are collinear.

Finally, consider the triangle PQR. We have shown that the line SP bisects the angle RPQ. Similarly, WQ bisects the angle PQR and ZR bisects the angle QRP. The three lines SP, WQ and ZR are therefore concurrent at the incentre of the triangle PQR.

Return to the problem

David's discovery is too nice to be just a coincidence, and this suggested the investigation of the problem described at the beginning of this article. From (1) and two similar equations, the values of α, β and γ in David's construction are

$$\alpha = \tfrac{1}{2}(c-b), \quad \beta = \tfrac{1}{2}(a-c), \quad \gamma = \tfrac{1}{2}(b-a); \tag{2}$$

if α, β and γ have these values then the three lines ST, VW and YZ are concurrent. Can we find any more triples α, β and γ with this property? Suppose α and β are given. Then we can draw the lines ST and WV and they will meet at some point, X say. For each choice of γ we can draw a line YZ and, as γ varies, we get a family of parallel lines, exactly one of which passes through X. It follows that the condition for concurrence must be a single equation involving α, β and γ.

Before reading on, take up pencil and paper and see if you can find the condition and prove that it works. To add to the suspense, this saga is continued later in this issue.

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MATHEMATICS (BE IN IT) COMPETITION

Competition Number 4

This term's competition is proposed by Andrew Jenkins (Year 7, North Sydney Boys' High School).

Even when using a modern electronic calculator, we still seem to make mistakes in our work, and curse ALFRED (A Little Flaming Ridiculous Electronic Device), the calculator, as much as we like, the chances are still 99.999% that it was our mistake caused by an error in the input.

Now here at last is a way in which you can check which function you called for. Simply do the following operation after setting the calculator to display two decimal places, and then read the display upside down:

$$\log 1174897551.$$

Now set the calculator to display zero decimal places and calculate these quantities:

$$5660 - 123$$

$$e^{1.098}$$

For this last example, hold the display right way up:

$$14 \text{ by } 6.$$

See if you can find some more examples like this.

Prizes of \$3, \$2 and \$1 will be awarded for the best entries. Entries must reach the Editor by 1st July, 1980, and the prize winners will be announced in Parabola, Volume 16, Number 3. Please make sure that your entry bears your name, address, year and school.

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