PROBLEM 415 RE-SOLVED

This rather tricky problem appeared in Volume 15, Number 1:

Thirty-two counters are placed on a chess-board so that there are four in every row and four in every column. Show that it is always possible to select eight of them so that there is one of the eight in each row and one in each column.

The solution in Volume 15, Number 3, was short and sweet and, correspondingly, unmotivated. Here is a less ad hoc solution, illustrating the usefulness of an appeal to graph theory.

Consider Figure 1 which shows two sets of points labelled 1 to 8 and 1′ to 8′ and connected by lines. A line represents a counter and we draw a line from point i in the first set to point j′ in the second exactly when there is a counter in the square (i,j) of the chess-board. Since there are exactly four counters in every row and every column of the chess-board, we have exactly four lines meeting at every point in both sets. We wish to show that, however these lines happen to be arranged, we can find eight of them which pair off the points in the two sets. Now we can relabel the points in both sets as we choose, so we are trying to show that, after a suitable relabelling, there is a set of eight lines connecting i in the first set with i′ in the second, as in Figure 2.

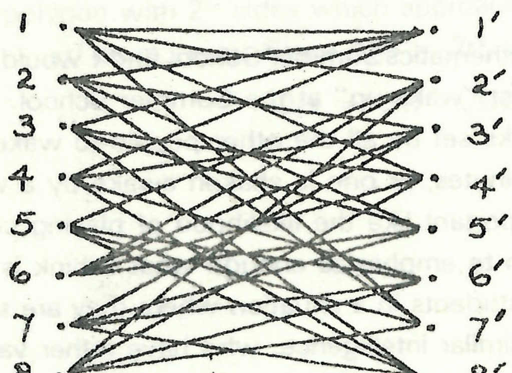

Figure 1

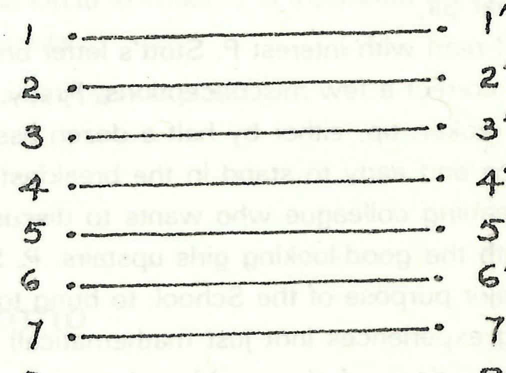

Figure 2

Suppose we have succeeded in relabelling the points so that i is connected to i′ for i = 1, 2, 3, ..., k with k < 8. (See Figure 3.) We wish to show that the labelling can be extended, perhaps altering the points already relabelled, so that k + 1 points are paired off in the two sets. It will follow that the number of points paired off can eventually be increased to 8, and the problem will be solved.

Choose any (k + 1)-th point in the first set. If it is connected to one of the 8 − k points left over in the second set, we can relabel that point (k + 1)′ and our task is accomplished. In fact, we can finish off in this way by choosing one of the unused points in the first set as the (k + 1)-th point unless all lines from the points k + 1, k + 2, . . ., 8 are connected to points j' with j ≤ k. In this case, it follows that all lines from the points (k + 1)', (k + 2)', . . ., 8' must run to points j with j ≤ k.

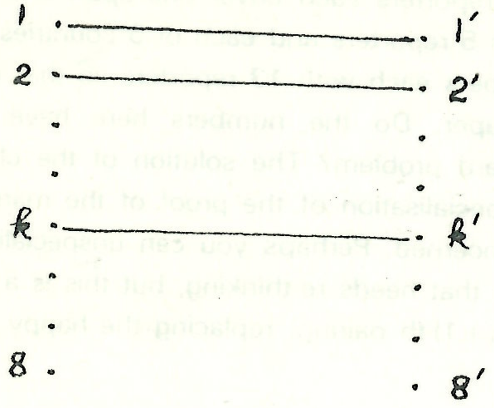

Figure 3

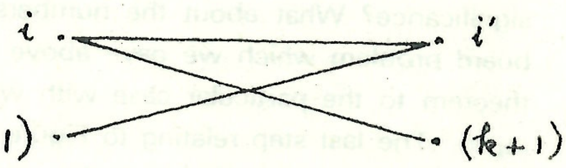

Figure 4

Suppose we are in the unfavourable case just described. The point k + 1 is connected to four of the points 1', 2', . . ., k' and the point (k + 1)' is connected to four of the points 1, 2, . . ., k. Since k ≤7, we can find some i so that k + 1 is connected to i' and (k + 1)' is connected to i. (See Figure 4.) But now if we interchange the labels i' and (k + 1)', we have achieved our goal.

Now you may be wondering what all this is about. If so, read on. Since the language of counters and chess-boards does not lend itself readily to dramatic insights, let us re-examine Figure 1 in somewhat more colourful terms. Let us call the points 1 to 8 men and the points 1' to 8' women; the lines from the i-th man lead to the women he is willing to marry. (It is possible that this interpretation is just a little dated.) The sixty-four million dollar question is whether every man can be married off to a woman on his list of desirable partners. In more prosaic terms, this amounts to finding eight lines in Figure 1 which pair off the points, so it is exactly the same problem as before. Suppose all the marriages can be arranged as required. If we take any k lists of desirable partners, they must contain at least k different names between them. (For, if they didn't, we would have k men nominating fewer than k partners and there would be no way of satisfying all of them.) It is rather surprising that this obvious condition is all we need in order to solve the marriage problem.

The marriage theorem. Consider a set of men and a set of women. Each man makes a list of the women he is willing to marry. Then, each man can be married to a woman on his list if and only if

(*) for every value of k, any k lists contain at least k different names between them.

Let us see how the marriage theorem solves the problem of the counters on the chess-board. Here, we have eight men and eight women, each man lists exactly four women he is willing to marry and each woman appears on exactly four lists. If we take any k lists, we get altogether 4k names, some of which may be repeated, but since no name can appear more than four times, there must be at least k different names. Thus our condition (*) is satisfied and we can find a pairing which solves the original problem.

The marriage theorem is often useful in matching problems. Here is an example which is a slight twist on the problem considered above: 65 newspaper reporters each cover one sport and one foreign country in such a way that each of 13 sports has 5 reporters and each of 5 countries has 13 reporters. Show that it is possible to staff 5 newspapers each with 13 reporters so that each sport and each country is covered by each newspaper. Do the numbers here have any significance? What about the numbers in the chess-board problem? The solution of the chess-board problem which we gave above is essentially a specialisation of the proof of the marriage theorem to the particular case with which we were concerned. Perhaps you can unspecialise it again. (The last step relating to Figure 4 is the only part that needs re-thinking, but this is a little tricky. The condition (*) has to be used to obtain the (k + 1)-th pairing, replacing the happy accident that $4+4 > 7$.)

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A NICE INTEGRAL

$$3 \int (\text{ice})^2 \, d(\text{ice}) = \text{ice-cube} + c = \text{ice-berg}$$

— I. Woodhouse, Marsden High School.

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COMMON SENSE

For integers a and b, the symbol (a,b) denotes the greatest common divisor of a and b, that is the largest integer which divides both a and b. (For example, (14,15) = 1 and (28,36) = 4.) The problems that follow ask you to simplify some horrendous expressions involving greatest common divisors. Hint: the problems have an *artistic* solution (which will be revealed in the next issue).

(1) Show that $(a/(a,c),\ b/(b,c)) = (a,b)/(a,b,c)$.

(2) Simplify $(a/(a,b,c),\ b/(b,c))$.

(3) Simplify $(a(b,c,d)/(a,c)\ (a,d),\ b(b,c,d)/(b,c)\ (b,d))$.

— H. Nayna, Year 3, University of New South Wales

THE ETERNAL TRIANGLE (CONTINUED FROM PAGE 12)

The dénouement

Let ABC be a triangle with sides BC = a, CA = b and AB = c. Measure off distances AS = α, BV = β and CY = γ along AB, BC and CA and distances AT = $-\alpha$, BW = $-\beta$ and CZ = $-\gamma$ along AC, BA and CB, as described earlier. (See Figure 5.) I assert that the lines ST, VW and YZ are concurrent if and only if

$$a\alpha + b\beta + c\gamma = 0. \tag{3}$$

(Note that this includes David McGrath's theorem because the values of α, β and γ given in (2) satisfy our equation (3).)

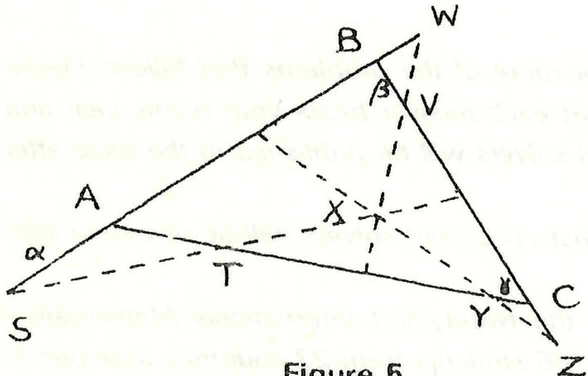

Figure 5

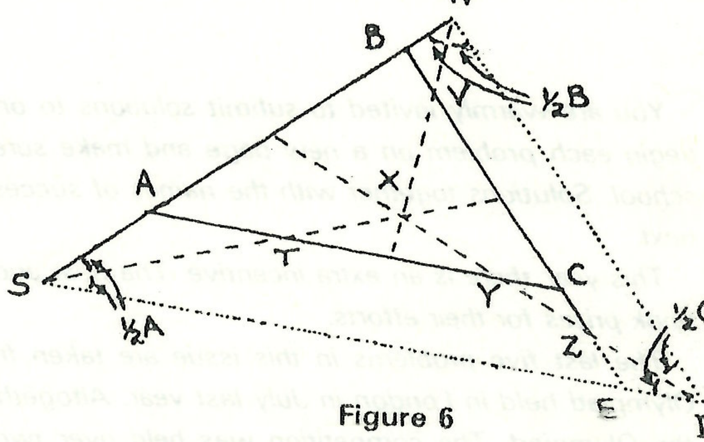

Figure 6

Now let us prove that (3) is satisfied if and only if our three lines are concurrent. Consider Figure 6 in which we have drawn SD parallel to AC and WD parallel to BC. Then ST bisects the angle DSW and WV bisects the angle SWD and these two bisectors meet at the incentre X of triangle SWD. Moreover, YZ is parallel to the bisector of the angle WDS. If we use again the fact that the angle bisectors of a triangle are concurrent, we see that YZ goes through X if and only if YZ actually is the bisector of angle WDS, as shown in Figure 6. So all we have to do to finish things off is to calculate the length of CY or CZ in Figure 6. We can do this with the aid of a little trigonometry. Our calculations are based on Figure 6 where $\alpha < 0$ and $\beta, \gamma > 0$; only minor sign changes are required to deal with other possibilities. The perpendicular distance between AC and SD is $-\alpha \sin A$ (remember that α is negative), so

$$DY = -\alpha \sin A / \sin \tfrac{1}{2}C.$$

Similarly,

$$DZ = \beta \sin B / \sin \tfrac{1}{2}C,$$

so

$$YZ = -(\alpha \sin A + \beta \sin B)/\sin \tfrac{1}{2}C.$$

But YZ is the base of the isosceles triangle CYZ with base angles ½C, so

$$\gamma = CY = -(\alpha \sin A + \beta \sin B)/2 \sin \tfrac{1}{2}C \cos \tfrac{1}{2}C = -(\alpha \sin A + \beta \sin B)/\sin C,$$

that is

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0. \tag{4}$$

Finally, to get (3), we use the fact that the area of the triangle ABC is

$$\triangle \text{ (say)} = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C,$$

and this enables us to eliminate the sines in (4). This completes the proof.