PROBLEM SECTION

You are warmly invited to submit solutions to one or more of the problems that follow. Please begin each problem on a new page and make sure that each answer bears your name, year and school. Solutions together with the names of successful solvers will be published in the issue after next.

This year there is an extra incentive. The best and most consistent solvers will be rewarded with book prizes for their efforts.

The last five problems in this issue are taken from the twenty-first International Mathematical Olympiad held in London in July last year. Altogether, 166 students from 23 countries took part in the Olympiad. The competition was held over two days and consisted of two four-hour papers, each containing three questions. Problems 450 and 451 are from the first paper and problems 452, 453 and 454 are from the second. Overall, the candidates scored 27% on problem 450, 44% on problem 451, 73% on problem 452, 56% on problem 453 and 56% on problem 454, so the problems are not impossible. In fact, four candidates scored full marks on all six questions. Problems 448 and 449 were amongst the problems suggested for the Olympiad, but were not used.

Happy puzzling.

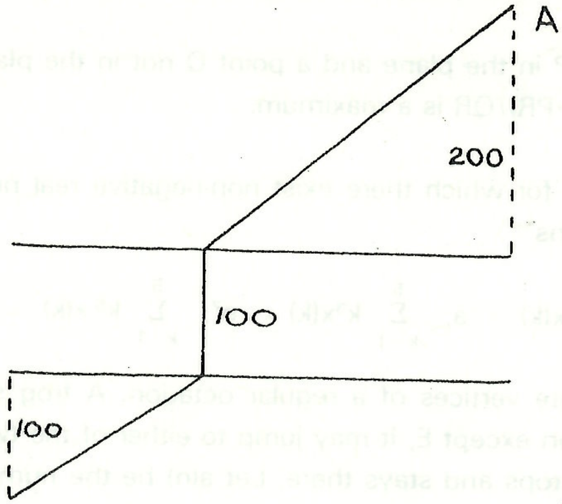
441. Prove that the number 1111 . . .11, consisting of 91 ones, is a composite number.

442. Factorise the polynomial $x^8 + x^4 + 1$ into factors of at most the second degree.

443. None of the numbers a, b or c is zero and each is a root of the equation $x^3 - ax^2 + bx - c = 0$. Find a, b and c.

444. Prove that if the sum of the positive numbers a, b and c is equal to 1, then $a^{-1} + b^{-1} + c^{-1} \geqslant 9$.

445. A river 100m wide runs due east-west. Points A and B are on opposite sides of the river and at respective distances of 200m and 100m from its banks. B is 400m further west than A. A road and bridge joining A and B is to be constructed subject to the condition that the bridge must cross the river perpendicularly. What is the shortest possible total length of road and bridge which will join the points? Prove your answer.



446. A rectangle is drawn so that its four vertices lie on the perimeter of a given acute-angled triangle. Find the locus of the centre of the rectangle as it moves subject to these constraints.

447. (i) If AB and CD are line segments of equal length ℓ which do not intersect, prove that at least one of AC, AD, BC and BD has a length greater than ℓ.

(ii) Let S be a set of n distinct points in the plane. Consider the $\frac{1}{2}n(n-1)$ line segments connecting all possible pairs of points of S. Any one of the longest of these line segments is called a diameter of S. There may be several diameters; for example, if S consists of the four points at the vertices of a rectangle, both diagonals are diameters. However, by (i), any two diameters must have a point in common.

Prove that a set of n points has at most n diameters.

(iii) Show that for any $n \geqslant 3$, there exists a set of n points having exactly n diameters.

448. Let S be the area of the parallelogram OABC. Prove that $S^3 \leqslant (3\sqrt{3}/8)\, OA^2 \cdot OB^2 \cdot OC^2$.

449. Prove that every regular polygon having an even number of sides can be dissected into lozenges. (A lozenge, or rhombus, is a quadrilateral whose four sides are all of the same length.)

450. Let p and q be integers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

451. A prism with pentagons A(1) A(2) A(3) A(4) A(5) and B(1) B(2) B(3) B(4) B(5) as top and bottom faces is given. Each side of the two pentagons and each of the line segments A(i) B(j), for all i,j = 1, 2, 3, 4, 5, is coloured either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been coloured has two sides of different colours. Show that all ten sides of the top and bottom faces of the prism are the same colour.

452. Given a plane, a point P in the plane and a point Q not in the plane, find all points R in the plane such that the ratio $(QP+PR)/QR$ is a maximum.

453. Find all real numbers a for which there exist non-negative real numbers x(1), x(2), x(3), x(4) and x(5) satisfying the equations

$$\sum_{k=1}^{5} kx(k) = a, \quad \sum_{k=1}^{5} k^3x(k) = a^2, \quad \sum_{k=1}^{5} k^5x(k) = a^3.$$

454. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a(n) be the number of distinct paths of exactly n jumps ending at E.

Prove that $a(2n-1) = 0$ and $a(2n) = \{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}\}/\sqrt{2}$, for n = 1, 2, 3, ...

(A path of n jumps is a sequence of vertices (P(0), P(1), . . ., P(n)) such that

(i) P(0) = A, P(n) = E,

(ii) P(i) $\neq$ E for $0 \leqslant i \leqslant n-1$, and

(iii) P(i) and P(i+1) are adjacent for $0 \leqslant i \leqslant n-1$.)

SOLUTIONS TO PROBLEMS FROM VOLUME 15, NUMBER 2

417. Let a and b be integers. Show that $10a+b$ is a multiple of 7 if and only if $a-2b$ is also.

Solution.

Note that

$$10a + b = 10(a-2b) + 21b \tag{1}$$

and

$$a - 2b = -2(10a+b) + 21a. \tag{2}$$

If 7 divides $a-2b$, then both terms on the right side of (1) are multiples of 7, whence $10a+b$ is also. Similarly, from (2), if $10a+b$ is a multiple of 7, so is $a-2b$.

Variations on the above were supplied by A. Choy (Trinity Grammar), D. Everett (Kotara High School), P. Rider (St. Leo's College), K. Svendsen (Busby High School), S. Tolhurst (Springwood High School), J. Tually (Sydney Grammar), S.S. Wadhwa (Ashfield Boys' High School), R. Wilson