451. A prism with pentagons A(1) A(2) A(3) A(4) A(5) and B(1) B(2) B(3) B(4) B(5) as top and bottom faces is given. Each side of the two pentagons and each of the line segments A(i) B(j), for all i,j = 1, 2, 3, 4, 5, is coloured either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been coloured has two sides of different colours. Show that all ten sides of the top and bottom faces of the prism are the same colour.

452. Given a plane, a point P in the plane and a point Q not in the plane, find all points R in the plane such that the ratio (QP + PR)/QR is a maximum.

453. Find all real numbers a for which there exist non-negative real numbers x(1), x(2), x(3), x(4) and x(5) satisfying the equations

$$\sum_{k=1}^{5} kx(k) = a, \quad \sum_{k=1}^{5} k^3x(k) = a^2, \quad \sum_{k=1}^{5} k^5x(k) = a^3.$$

454. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a(n) be the number of distinct paths of exactly n jumps ending at E.

Prove that $a(2n-1) = 0$ and $a(2n) = \{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}\}/\sqrt{2}$, for n = 1, 2, 3, ...

(A path of n jumps is a sequence of vertices (P(0), P(1), . . ., P(n)) such that

(i) P(0) = A, P(n) = E,

(ii) $P(i) \neq E$ for $0 \leqslant i \leqslant n-1$, and

(iii) P(i) and P(i + 1) are adjacent for $0 \leqslant i \leqslant n-1$.)

SOLUTIONS TO PROBLEMS FROM VOLUME 15, NUMBER 2

417. Let a and b be integers. Show that 10a + b is a multiple of 7 if and only if a − 2b is also.

Solution.

Note that

$$10a + b = 10(a-2b) + 21b \tag{1}$$

and

$$a - 2b = -2(10a+b) + 21a. \tag{2}$$

If 7 divides a − 2b, then both terms on the right side of (1) are multiples of 7, whence 10a + b is also. Similarly, from (2), if 10a + b is a multiple of 7, so is a − 2b.

Variations on the above were supplied by A. Choy (Trinity Grammar), D. Everett (Kotara High School), P. Rider (St. Leo's College), K. Svendsen (Busby High School), S. Tolhurst (Springwood High School), J. Tually (Sydney Grammar), S.S. Wadhwa (Ashfield Boys' High School), R. Wilson

(The King's School) and O. Wright (Davidson High School).

This problem is the basis of a rather curious test for divisibility by 7 discovered by A. Zbikovski, a Russian, in 1861. To see if a number is divisible by 7, remove the last digit, double it, subtract it from the truncated original number and continue doing this until only one digit remains. The original number is divisible by 7 if and only if the final digit is 0 or 7. For example, is 123456 divisible by 7?

```
123456
    12
------
 12333
     6
------
  1227
    14
------
   108
   16
------
  - 6
```

Since −6 isn't divisible by 7, neither is 123456. Can you see why this works? We do not claim it is quicker than dividing the original number by 7, only more interesting.

418. Two classes organised a party. To meet the expenses, each pupil of class A paid \$5 and each pupil of class B paid \$3. If the pupils of class A had paid all the expenses, they would have paid \$k each. At a second similar event, the pupils of class A paid \$4 each and those of class B paid \$6 each, and the total sum was the same as if each pupil in class B had paid \$k. Find k. Which class had more pupils?

Solution.

Let a and b be the numbers of pupils in classes A and B respectively. From the information concerning the first party, $5a + 3b = ka$, that is

$$(5-k)a + 3b = 0. \quad (1)$$

Similarly, from the second event,

$$4a + (6-k)b = 0. \quad (2)$$

From (1) and (2),

$$(5-k)/3 = 4/(6-k),$$

both ratios being equal to $-b/a$. This gives $k^2 - 11k + 18 = 0$, which has as solutions $k = 2$ and $k = 9$. However, $k = 2$ cannot be relevant to the problem since it yields $b/a = -1$, an obvious impossibility. Hence $k = 9$ and $b/a = 4/3$ so that class B has more pupils than class A.

This problem was solved by A. Choy (Trinity Grammar), K. Lim (St. Ignatius' College), P. Rider (St. Leo's College), K. Svendsen (Busby High School), S. Tolhurst (Springwood High School), J. Tually (Sydney Grammar), S.S. Wadhwa (Ashfield Boys' High School), R. Wilson (The King's

School) and O. Wright (Davidson High School). J. Tually noticed a quick way to deal with equations (1) and (2): adding the equations gives $9(a+b) = k(a+b)$ whence $k = 9$, as before. But what has happened to the solution $k = 2$?

419. Write on a large blackboard the numbers 1, 2, 3, . . ., 1979. Erase any two of the numbers and replace them by their difference. Repeat this process until only a single number is left on the board. Prove that this number is even.

Solution.

In each step of the process, the number of odd numbers present either remains unchanged, or decreases by 2. (The latter possibility occurs if both numbers erased are odd, their difference then being an even number.) Since the number of odd numbers at the start is $\frac{1}{2}(1 + 1979) = 990$, an even number, the number of odd numbers on the blackboard remains even throughout the whole operation. Thus, when there is only one number left, it cannot be an odd number.

The above argument was presented very clearly by A. Johnston (Ignatius Park College), J. Tually (Sydney Grammar), S.S. Wadhwa (Ashfield Boys' High School), and R. Wilson (The King's School), and quite satisfactorily by K. Lim (St. Ignatius' College) and K. Svendsen (Busby High School).

420. King Arthur's knights arrange a tournament. After it is all over, the king notices that to every two knights, there is a third one who has vanquished both. How many knights (at least) must have taken part in the tournament?

Solution.

At least 7 knights must have taken part. To show that 6 knights are insufficient, we can argue as follows. Label any 2 of the knights A and B. Let C be a knight who beat A and B, D be a

	A	B	C	D	E	F	G
A		0	0	0	1	1	1
B	1		0	1	0	1	0
C	1	1		0	0	0	1
D	1	0	1		1	0	0
E	0	1	1	0		1	0
F	0	0	1	1	0		1
G	0	1	0	1	1	0	

Figure 1

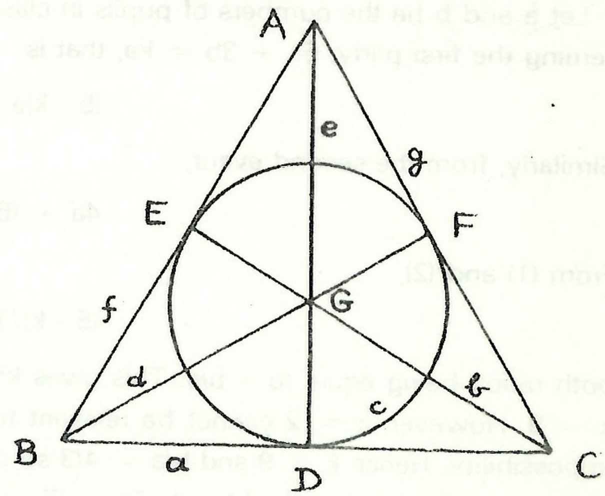


Figure 2

knight who beat B and C, E be a knight who beat C and D and, finally, F be a knight who beat C and E. This gives 6 different knights. If there are no more knights, we see that D is the only one who could have beaten C and F, and A is the only one who could have beaten D and F, and so B must have been the conqueror of A and F. But now it is impossible to find anyone who beat both B and E, since B beat A and F and E beat C and D.

On the other hand, the tournament score-card in Figure 1 shows that the requirements can be met by 7 knights. (For example, row D of the score-card means that D beat A, C and E, and D was beaten by B, F and G.) In the rather different form shown in Figure 2, this is a well-known configuration. Figure 2 has 7 points, labelled A to G, and 7 edges, labelled a to g, including the circular edge c, and each pair of edges has a common point. In our problem, we interpret this as follows. The points B, C, D on the edge a are the knights who beat A and the point C common to the edges a and b is the knight who beat A and B, and so on. For reasons which we cannot go into here, Figure 2 is known as the projective plane of order 7. Observe that what we needed to solve the problem was a configuration of points and lines so that every pair of points is joined by a line and every pair of lines meets in at least one point. Figure 2 is the smallest system with these properties; it represents a rather peculiar sort of geometry.

421. In the sequence 19796..., each digit after 6 is the last digit of the sum of the preceding four digits. (Thus, the next digit is 1 since $9 + 7 + 9 + 6 = 3\underline{1}$.) Show that ...1979... turns up again in the sequence, but that ...1980... never occurs at all.

Solution from R. Wilson (The King's School), with additional explanations.

If we take account only of the parity of the successive digits and write O for an odd digit and E for an even digit, the sequence becomes

O O O O E O O O O E O O O O E ...,

that is, there are blocks of 4 odd digits separated by a single even digit throughout the sequence. (This follows since the sum of 4 odd numbers ends in an even digit, and the sum of 3 odd numbers and an even number ends in an odd digit.) Hence ...1980... cannot occur, since it contains consecutive even digits.

Sooner or later, the sequence of digits must recur. There are only $5^4 = 625$ different blocks of 4 odd digits. Hence, if one looks at the first 626 blocks OOOOE, there must be two with the same block of odd digits. But then the even digits at the end must also be the same, and then the following odd digits as well. Thus the sequence has begun to cycle. Moreover, the rule of construction determines uniquely the predecessor of any block of 4 digits. (For example, if ...x3561... occurs, then x must be 7 since $x + 3 + 5 + 6 = 1 + 10n$ for some non-negative integer n.) Thus the cyclic behaviour of the sequence must extend right back to the beginning and, in particular, 1979 must occur again later in the sequence.

The first part of the problem was also solved by J. Tually (Sydney Grammar).

422. The heptagon ABCDEFG is inscribed in a circle (that is, all of its vertices A, B, . . ., G are on the circle) and three of its angles are 120°. Prove that the heptagon has two equal sides.

Solution.

First observe that, in Figure 1, if angle APB is 120°, then the arc AB is a third of the circumference of the circle. (Note first that the reflex angle AOB is twice 120°, that is 240°, whence the arc APB subtends an angle of 120° at the centre.) It follows that two of the 120° angles in the heptagon must have been neighbouring angles. For otherwise, as in Figure 2, the arcs AB, BC and CD would already account for the whole circumference of the circle, so that D must coincide with A, that is the heptagon collapses into a hexagon. Thus we conclude that two neighbouring angles of the heptagon, at the vertices X and Y in Figure 3, must be equal. Now the portion WXYZ of the heptagon is symmetrical about OM, the perpendicular bisector of XY and consequently the sides WX and YZ are equal in length.

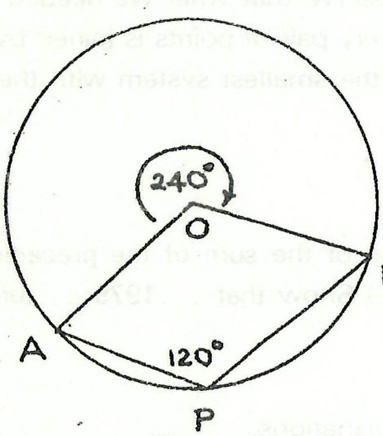


Figure 1

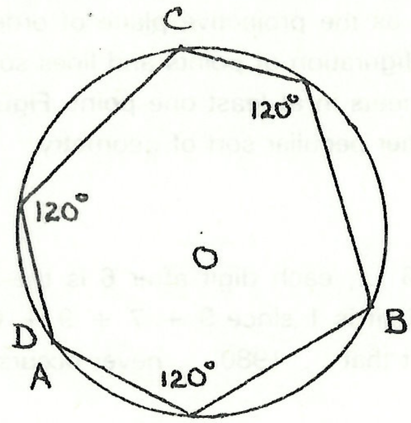


Figure 2

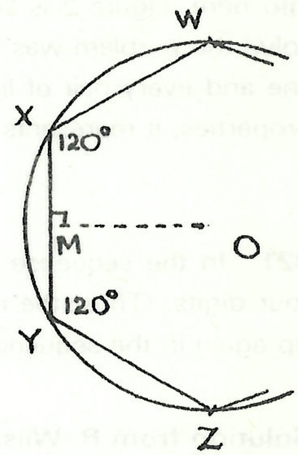


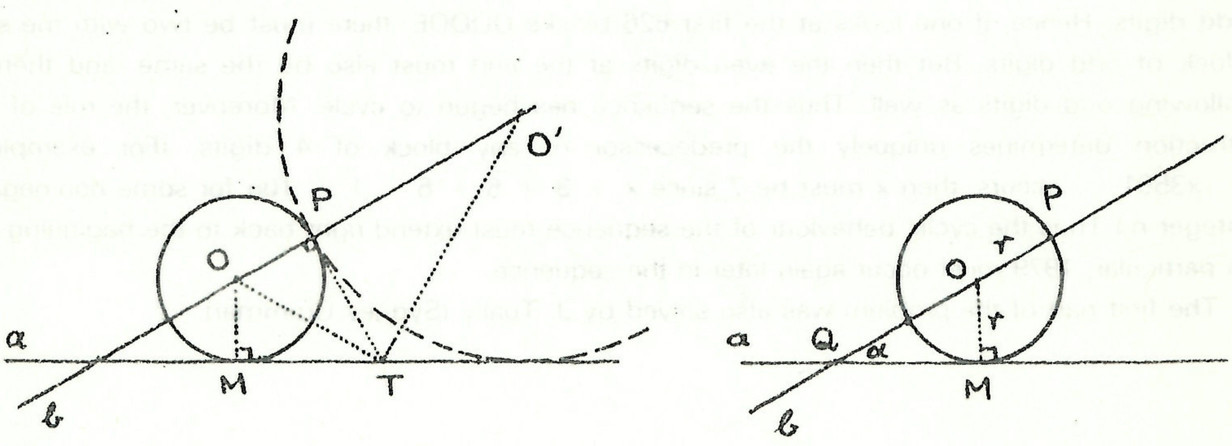
Figure 3

Good solutions were received from K. Svendsen (Busby High School) and S.S. Wadhwa (Ashfield Boys' High School).

423. Given two intersecting straight lines a and b and a point P on b, show how to construct a circle whose centre is on b and which passes through P and touches a.

Solution I from A. Jenkins (North Sydney Boys' High School).

In fact, there are two possible circles.



Draw a line PT perpendicular to b, to intersect a at T. Draw the bisectors of the angles at T, intersecting b at O and O′. Then O and O′ are the centres of the required circles. For example, if M is the foot of the perpendicular from O to a, then triangles OMT and OPT are congruent, so OM = OP, that is TM is tangential to the circle.

This construction was also given by K. Svendsen (Busby High School). A. Choy (Trinity Grammar) found a more complicated construction. J. Taylor (Woy Woy High School) gave a construction which works only for special positions of the lines a and b.

Solution II from T. Abberton (St. Paul's College, Bellambi).

We shall calculate where to put the centres of the circles on the line b by using a little trigonometry. Consider the circle in the second figure with centre O and radius r, say, which passes through P and touches the line a at M. We want to calculate the distance OQ in terms of the given distance PQ and the given angle α between the lines a and b. Now

$$PQ = r + OQ \quad \text{and} \quad r = OQ \sin \alpha,$$

so

$$OQ = PQ/(1 + \sin \alpha).$$

This formula gives the centre of one of the circles. The centre of the other circle is at the point O′ on b where

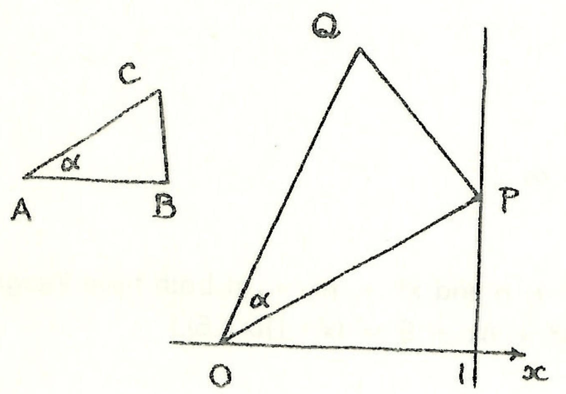
$$O'Q = PQ/(1 - \sin \alpha).$$

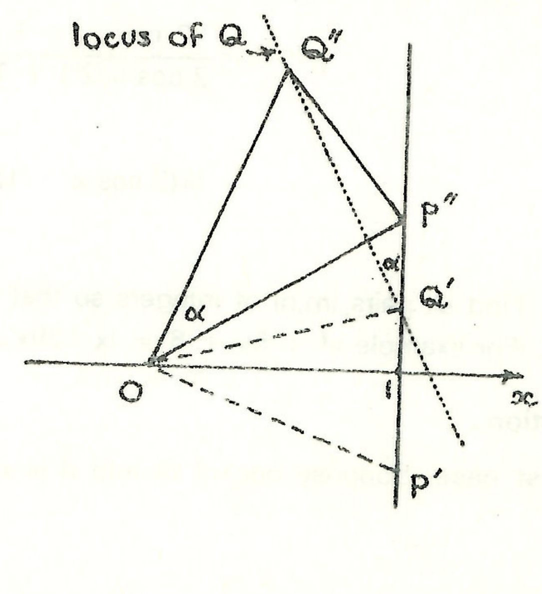
Can you see why?

424. A triangle ABC is given in the x-y plane. Now, O is the origin, the point P moves along the line x = 1 and the point Q is determined so that the triangles ABC and OPQ are similar (that is, angle QOP = angle CAB and angle QPO = angle CBA). Describe the motion of Q as P moves.

Solution.

The right hand figure shows two positions of triangle OPQ: triangle OP′Q′ is the position in which P′ and Q′ both lie on the line x = 1, and triangle OP″Q″ is any other position. Since angle





OQ''P'' = angle OQ'P', it follows that OQ'P''Q'' is a cyclic quadrilateral. (In our figure, the opposite angles OQ''P'' and OQ'P'' are supplementary. In other figures with P'' and P' on the same side of Q' on the line $x = 1$, the equal angles are both subtended by the chord OP''.) Again referring to the figure, angle Q''Q'P'' = angle Q''OP'', since both angles are subtended by the chord Q''P'', and angle Q''OP'' = angle CAB by construction. Thus Q'' always lies on the straight line through Q' making the angle CAB with the line $x = 1$. This line is the required locus of Q.

No correct solutions were received, but S.S. Wadhwa (Ashfield Boys' High School) knew that the locus of Q was a straight line.

425. Show that $2\cos x + 1 = 4\cos^2 \frac{1}{2}x - 1$. Find

$$\lim_{n\to\infty} (2\cos(x/2) - 1)(2\cos(x/2^2) - 1)\ldots(2\cos(x/2^n) - 1).$$

Solution from S.S. Wadhwa (Ashfield Boys' High School).

To prove the identity, note that

$$2\cos x + 1 = 2(2\cos^2 \tfrac{1}{2}x - 1) + 1 = 4\cos^2 \tfrac{1}{2}x - 1.$$

Now, to find the limit, observe that the identity gives

$$2\cos x + 1 = (2\cos \tfrac{1}{2}x + 1)(2\cos \tfrac{1}{2}x - 1),$$

that is

$$2\cos \tfrac{1}{2}x - 1 = (2\cos x + 1)/(2\cos \tfrac{1}{2}x + 1).$$

Similarly,

$$2\cos \tfrac{1}{4}x - 1 = (2\cos \tfrac{1}{2}x + 1)/(2\cos \tfrac{1}{4}x + 1),$$

and, in general,

$$2\cos(x/2^n) - 1 = (2\cos(x/2^{n-1}) + 1)/(2\cos(x/2^n) + 1).$$

From this,

$$(2\cos(x/2) - 1)(2\cos(x/2^2) - 1)\ldots(2\cos(x/2^n) - 1)$$

$$= \frac{2\cos x + 1}{2\cos(x/2) + 1}\cdot\frac{2\cos(x/2) + 1}{2\cos(x/2^2) + 1}\cdot\ldots\cdot\frac{2\cos(x/2^{n-1}) + 1}{2\cos(x/2^n) + 1}$$

$$= \frac{2\cos x + 1}{2\cos(x/2^n) + 1}$$

$$\to \tfrac{1}{3}(2\cos x + 1), \text{ as } n \to \infty.$$

426. Find all pairs (m,n) of integers so that $x^2 + mx + n$ and $x^2 + nx + m$ both have integer roots. (For example $x^2 + 5x + 6 = (x+2)(x+3)$ and $x^2 + 6x + 5 = (x+1)(x+5)$.)

Solution.

First case. Suppose one of m and n is zero, say $m = 0$. Since $x^2 + n$ has two integer roots,

$-n$ must be a perfect square, that is $n = -k^2$ with $k = 0, 1, 2, \ldots$. This gives the quadratics $x^2 - k^2 = (x+k)(x-k)$ and $x^2 - k^2x = x(x-k^2)$.

Second case. Suppose $m = n$. Since $x^2 + mx + m$ has two integer roots, $m^2 - 4m = (m-2)^2 - 4$ must be a perfect square. In the list of squares $0, 1, 4, 9, \ldots$, the only two which differ by 4 are 0 and 4. Hence $m^2 - 4m = 0$, giving $m = 4$. (The solution $m = 0$ is covered by the first case.) So here we get the quadratic $x^2 + 4x + 4 = (x+2)^2$.

Third case. Suppose $m = -n$. The argument used in the second case shows that no new examples can appear here.

Fourth and last case. Suppose neither m nor n is zero and $m \neq \pm n$, say $|m| > |n|$. Let α and β be the integer roots of $x^2 + mx + n$, so that

$$|\alpha| + |\beta| \geqslant |\alpha+\beta| = |m| > |n| = |\alpha| \cdot |\beta| \qquad (1)$$

It follows that the smaller of $|\alpha|$ and $|\beta|$ must be 1. (For if both $|\alpha|$ and $|\beta|$ are greater than or equal to 2, then $|\alpha| \cdot |\beta| \geqslant |\alpha| + |\beta|$, contrary to (1).) Also, α and β must have the same sign, for otherwise $|\alpha+\beta| < |\alpha| \cdot |\beta|$, again contrary to (1). Thus $n = \alpha\beta$ must be a positive integer and $|m| = |\alpha| + |\beta| = n + 1$, so $m = \pm(n+1)$. Then

$$x^2 + mx + n = x^2 \pm (n+1)x + n = (x \pm 1)(x \pm n)$$

always has integer roots. Now we must consider $x^2 + nx + m = x^2 + nx \pm (n+1)$. With the lower sign, we get $x^2 + nx - (n+1) = (x-1)(x + (n+1))$, which always has integer roots. However, $x^2 + nx + (n+1)$ has integer roots only if $n^2 - 4(n+1) = (n-2)^2 - 8$ is a perfect square. In the list of squares $0, 1, 4, 9, \ldots$, the only two which differ by 8 are 1 and 9. Hence $(n-2)^2 = 9$, yielding $n = 5$. (The other root $n = -1$ is ruled out since n is positive.) Thus this case yields the quadratics

$$x^2 - (n+1)x + n = (x-1)(x-n), \quad x^2 + nx - (n+1) = (x-1)(x + (n+1)) \quad (n = 1,2,3,\ldots)$$

and

$$x^2 + 6x + 5 = (x+1)(x+5), \quad x^2 + 5x + 6 = (x+2)(x+3).$$

D. Everett (Kotara High School) found some of the above solutions.

427. The four aces, kings, queens and jacks are taken from a pack of cards and dealt to four players. Thereupon, the bank pays \$1 for every jack held, \$3 for every queen, \$5 for every king and \$7 for every ace. In how many ways can it happen that all four players receive equal payments (namely \$16)?

Solution.

First case. Suppose each player receives one ace, one king, one queen and one jack. There are 4.3.2.1 = 24 ways of allocating the aces to the four players, and similarly 24 ways of allocating the kings, queens and jacks. Hence there are 24.24.24.24 deals of this type.

Second case. Suppose two players receive a pair of aces and a pair of jacks and the other two each receive a pair of kings and a pair of queens. There are 3 different ways of separating the 4 aces into 2 pairs. For each pair of aces, there are 6 different pairs of jacks. Hence there are 3.6 = 18 different pairs of hands consisting of 2 aces and 2 jacks. Similarly, there are 18 different pairs of hands consisting of kings and queens. Thus there are 18.18 different ways of preparing the 4 hands, and then 4.3.2.1 ways of allocating them to the 4 players. Thus there are 18.18.24 different deals of this type.

Third case. Suppose one player receives an ace and 3 queens, a second receives a jack and 3 kings, a third receives one ace, one king, one queen and one jack, and the fourth receives 2 aces and 2 jacks. There are 4 ways of choosing the ace to be included in the first hand and 4 ways of choosing the queen to be excluded, giving 16 different hands of this type. Similarly, there are 16 different hands of the second type. Having already chosen the first two hands, there remain 3 ways of choosing an ace and 3 ways of choosing a jack for the third hand. This gives 16.16.9 ways of selecting the 4 hands and, as before, 24 ways of allocating the hands to the 4 players, yielding a total of 16.16.9.24 deals of this type.

Fourth case. Suppose one player receives 2 aces and 2 jacks, a second receives 2 kings and 2 queens, and the remaining players each receive one ace, one king, one queen and one jack. There are 6 ways of selecting the pair of aces, jacks, kings and queens for the first 2 hands. The remaining 2 aces having been dealt, there are 2 different ways of distributing the remaining 2 kings, 2 queens and 2 jacks, and finally 24 ways of giving the prepared hands to the players. This gives 6.6.6.6.2.2.2.24 deals of this type.

Adding the number of deals of the 4 types gives a total of 643,680. (This takes no account of the order in which each of the 4 players receives the cards in the hand dealt to him.) Since there are altogether $16!/(4!)^4 = 63{,}063{,}000$ different deals, the chance that each player receives \$16 is very slightly more than 1%.

K. Svendsen (Busby High School), J. Taylor (Woy Woy High School) and S.S. Wadhwa (Ashfield Boys' High School) both observed the 4 types of deals described above, but did not count the number of ways in which each type might arise.

428. Let n be an integer whose last digit is 7. Show that some multiple of n has no digit equal to zero.

Solution.

Although it appears innocent enough, this is a very difficult problem. The solution contains an idea that is often useful.

Consider the remainders when each of the numbers $10, 10^2, 10^3, \ldots, 10^n, 10^{n+1}$ is divided by n. As there are $n+1$ numbers in the list, but only n different remainders possible, at least two remainders must be equal; say these are the remainders from 10^s and 10^t with $s < t$. The difference of these two numbers is a multiple of n, that is, n divides $10^t - 10^s = 10^s(10^{t-s} - 1)$. Since the last digit of n is 7, neither 2 nor 5 is a factor of n, so we deduce that n divides $10^{t-s} - 1$. So $10^{t-s} - 1$ is a multiple of n and none of its digits is 0; in fact, $10^{t-s} - 1 = 999\ldots99$.

Solvers of earlier problems.

The following contribution was received too late for acknowledgement in the last issue:

K. Svendsen (Busby High School): an impressive solution to problem 416.