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A MATHEMATICAL CHRISTMAS

BY

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1. Trees

Most of us will have a tree in our homes over Christmas. To the mathematician, a tree is a special type of *graph*. In this context, a graph is simply a finite set of points in the plane, called *vertices*, with certain points joined to others by lines, called *edges*. So Figures 1 and 2 both give examples of graphs. Such configurations can arise, for example, with transport networks such as the London Underground system, where the vertices are stations and the edges are underground lines connecting various stations.

You may be able to spot a difference between the graphs in Figure 1 as opposed to those in Figure 2. In the graphs in Figure 2, it is possible to start at one of the vertices and travel along edges through distinct vertices so as to arrive back at your starting point (like travelling right round the Circle Line on the London Underground!). We say that the graphs in Figure 2 have *circuits*, whereas those in Figure 1 do not. The two graphs in Figure 1 are called *trees*, those in Figure 2 are not. Thus, mathematically at least, a tree is a graph with no circuits. (Even Figure 1(b) is a tree, 'well known for its bark', as Robin Wilson remarks in Reference 1!)

If we now count the numbers of vertices and edges of these various graphs, we see that Figure 1(a) has 17 vertices and 16 edges, Figure 1(b) has 11 vertices and 10 edges, Figure 2(a) has 17 vertices and 22 edges and Figure 2(b) has 11 vertices and 12 edges. Is it possible to tell whether a graph is a tree by comparing the number of its vertices with the number of its edges? If we assume that our graph is *connected* in the sense that it is possible to get from every

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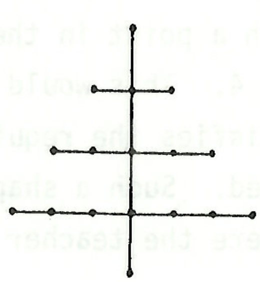
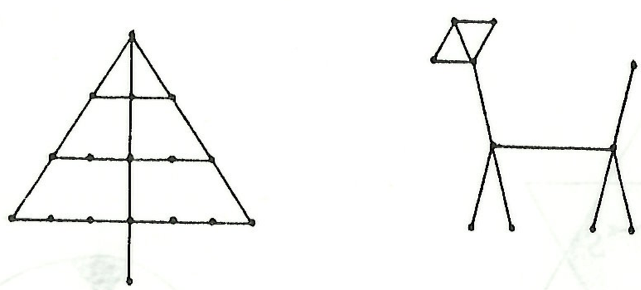


(a) (b)

Figure 1



(a) (b)

Figure 2

vertex to every other by travelling along edges, a reasonable assumption in the case, say, of an underground system, then the answer is 'yes'. It is not difficult to convince yourself that the trees are the graphs which have one edge fewer than they have vertices. All the rest have at least as many edges as vertices.

2. Stars

Your Christmas tree will probably have either a fairy or a star on top. What is a star? Mathematically, it is a geometrical figure which contains a point S with the property that every other point of the figure can be joined to S by a straight line lying wholly within the figure. For example, any point in the inner hexagon of Figure 3 can be taken as S. Thus a star-shaped room is one which has a point from which every other point of the room is visible. No doubt trainee teachers are advised to stand at such a point in the classroom! Another example of a star-shaped figure is in Figure 4. This would look rather odd on your Christmas tree, but it nevertheless satisfies the requirements for a star. On the other hand, Figure 5 is not star-shaped. Such a shape would be most inadvisable for a classroom, since there is nowhere the teacher could stand so as to see every student.

Here is a fascinating result about star-shaped regions in the plane. Suppose that, for every three points in the region, there is a point in the region from which all three points are visible. Then there is a point in the region from which *every* point is visible, i.e. the region is star-shaped. This is called *Krasnoselľskii's Theorem*.

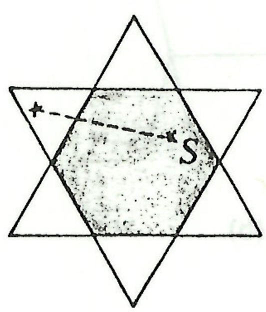


Figure 3

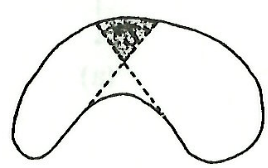


Figure 4

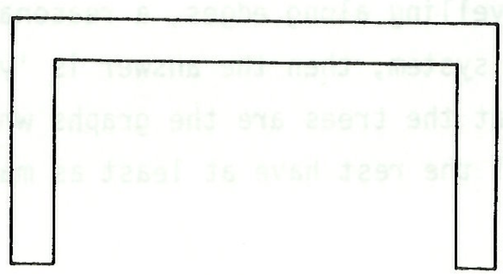


Figure 5

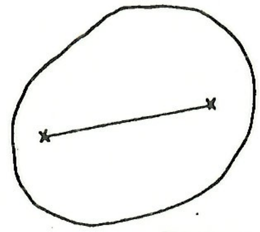


Figure 6

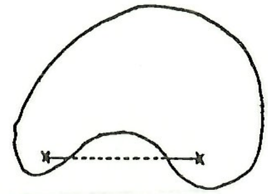


Figure 7

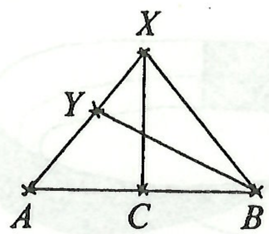


Figure 8

A *convex* figure is one such that every two points of it can be joined by a straight line lying wholly within the figure. Figure 6 is an example of a convex figure, whereas Figure 7 is not. A convex figure is star-shaped from every point of it.

In Figures 3 and 4, the points from which the figures are star-shaped are those in the shaded regions. We call these shaded regions the *kernels* of the figures. What property do these kernels possess? Why, they are convex. In 1912, a mathematician by the name of Brun proved that this is always the case, so that the kernel of a star-shaped figure is always convex.

We can quite easily give an argument to prove Brun's result. We refer to Figure 8. Let A, B be two points in the kernel of our star-shaped figure (whose boundary is now shown in Figure 8), and let C be any point on the line segment joining A, B. We must show that every point of the figure is visible from C, so consider any point X of the figure. We must show that the whole line segment CX lies in our figure. Now each point Y on the line segment AX lies in the figure (because A is in the kernel). Thus each line segment BY lies in the figure (because B is in the kernel). Since this is true for every Y on the line segment AX, the whole triangle ABX lies in the figure. In particular, the whole line segment CX lies in figure, so C really is in the kernel, as we claimed.

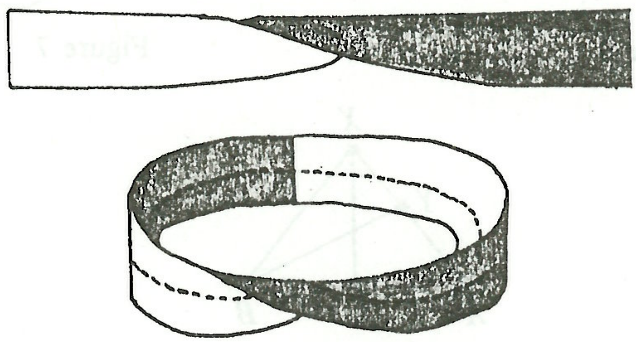


Figure 9

3. Decorations

We now describe a way of decorating your Christmas tree. Take a rectangular strip of paper, preferably brightly coloured, much longer than it is wide, and put a single twist in it. Now glue the two free ends together. We now have a *Möbius band*† (see Figure 9).

A Möbius band is a very curious mathematical object. For example, how many sides does it possess? Answer: only one. Try tracing round a side with a pencil as shown by the dotted line in Figure 9. You will eventually come back to your starting point. How many edges does the band possess? Answer: only one.

Now take a pair of scissors and cut your Möbius band down the middle. You will be amused at the result. Or you could cut it one-third of the way across. Your tree will end up festooned with exciting mathematical figures.

A mathematician confided
That a Möbius band is one-sided,
And you'll get quite a laugh
If you cut one in half,
For it stays in one piece when divided.‡

4. Fairy-tales

Christmas is a time for pantomimes and fairy-tales. Here is a mathematical fairy-tale. Take two Möbius strips. Each has only one edge. Bring the two edges together and sew the strips together. The result is a mathematical object called a *Klein bottle* after the German mathematician Felix Klein who invented it in 1882. We refer to a Klein bottle as a mathematical object because it cannot exist in three dimensions. In our drawing of it in Figure 10, it appears to pass through itself and therefore to have a circular hole in it (the dotted line in the figure). But for the mathematician this hole does not exist. The bottle has no rim and only one side. You would be ill-advised to try to use a Klein bottle to contain your Christmas drinks; nothing will stay in it because it has no inside or outside!

A mathematician named Klein
Thought the Möbius band was divine.
Said he, If you glue
The edges of two,
You'll get a weird bottle like mine.'‡

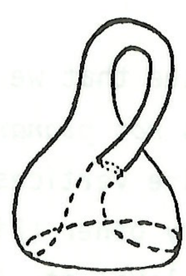


Figure 10

† A.F. Mobius was a German mathematician who in 1858 first considered such a band.

‡ See Reference 2.

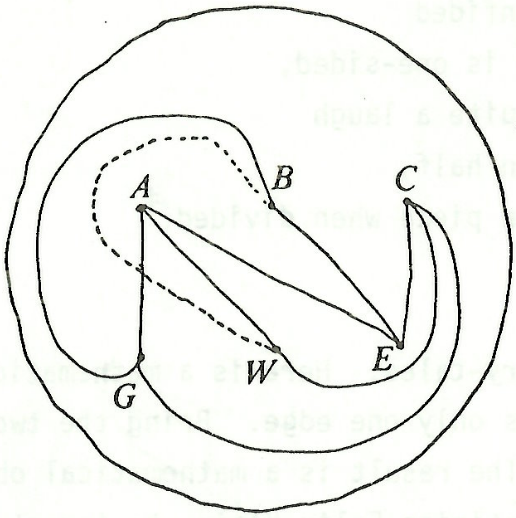


Figure 11

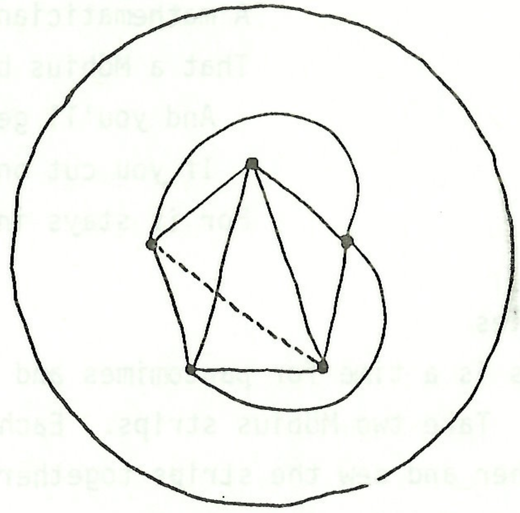


Figure 12

5. Balloons

No Christmas party is complete without balloons which, for our purposes, we may think of as spheres. We now describe a way of decorating a balloon.

The balloon represents the earth. Take a felt-tipped pen and mark three houses A, B, C. Depending on the size of the balloon these may have to be blobs. These three houses are to be connected to the gas, water and electricity services, denoted by three more blobs G, W, E. Thus each of A, B, C must be joined to each of G, W, E by a pipe or wire, which we draw with a line. The question is this: can this be done without these lines crossing? Try it on your balloon. (The problem is the same on a plain piece of paper, but less fun.) No matter how much you try, it cannot be done; two of the lines must cross (see Figure 11).

If we refer back to Section 1, we see that we have a graph with 6 vertices and 9 edges, and we say that this graph is *not planar*. Another example of a non-planar graph is obtained by marking five vertices on your balloon or piece of paper and joining every vertex to every other (Figure 12). We call this graph *the complete graph on 5 vertices*, and label it K_5 .

Is there a strange planet upon which we might stumble, hitch-hiking around the galaxy, which might allow us to join our houses to the services without the supply lines crossing? If there is a doughnut - (or swimming-ring-) shaped planet, then the answer is yes. This we have shown in Figure 13, where the dotted lines go through the hole of the doughnut and the dashed line goes round the back. Mathematicians for some reason prefer to speak of a *torus* rather than a doughnut! It is also possible to draw the graph K_5 on a doughnut/torus with no crossing edges (Figure 14).

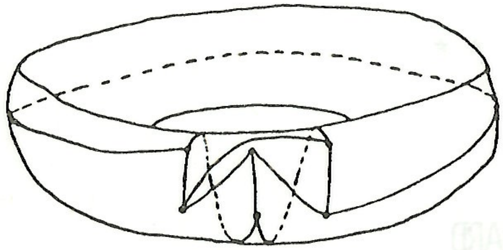


Figure 13

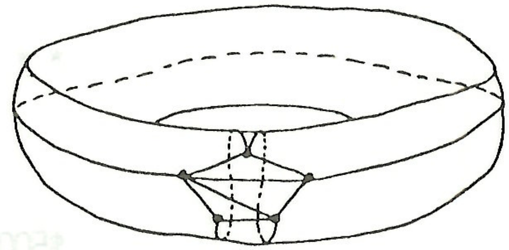


Figure 14

6. Food

This suggests an exciting addition to your Christmas party fare. Decorate doughnuts using piped icing with these graphs. Your friends will be fascinated! You can really go to town with this idea. You can also decorate your doughnut with K_6 and K_7. But try it with K_8 and it will not work. In fact, to draw K_8 without crossing edges you need a doughnut with two holes, shaped like a figure 8, called a *double torus* in the trade. And for K_9 you will need three holes! In general, for K_n you will need h holes, where h is the smallest integer greater than or equal to $^1/_{12}(n - 3)(n - 4)$. This was proved after a long and difficult struggle by two American mathematicians, Ringel and Youngs, in 1968. The corresponding result for the graph with m houses and n services is that you need p holes, where p is the smallest integer greater than or equal to $\frac{1}{4}(m - 2)(n - 2)$. Thus $K_{3,3}$ above needs one hole, as we saw, but $K_{4,5}$ needs two. (See reference 1, p.69.)

With these suggestions, next Christmas could be the best ever!

References

1. Robin J. Wilson, *Introduction to Graph Theory*, 2nd edn. (Longman, London, 1979).

2. Albert W. Tucker and Herbert S. Bailey Jr., 'Topology', *Scientific American*,

January 1950. Reprinted in *Mathematics in the Modern World* (W.H. Freeman, San Francisco, 1968).

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$500 REWARD

Professor Michael Cowling of the School of Mathematics of the University of New South Wales is offering $500 reward for information leading to the solution of the dangerous problem described below. The School of Mathematics is offering consolation prizes for interesting contributions towards the solution of the problem.

PROBLEM: Let A be a region in the plane of area one; A may have several components, and need not be convex, so that A may have holes. Given a positive integer n, let A_n be the region consisting of all points in the plane which are centres of circles of whose circumference at least $(1/n)$th lies in A. How fast does the area of A_n grow as n increases? For instance, is it true the area of A_n is less than $1000n^2$, or $1{,}000{,}000n^3$, no matter what the shape of A may be? In order to be eligible for the full reward, it is not sufficient to consider only particular regions A.

Solutions, together with the name, address, and age of the solver, must be sent to Professor M. Cowling, School of Mathematics, University of New South Wales, P.O. Box 1, Kensington 2033, and must arrive by December 1, 1984. A panel of judges from the School of Mathematics will examine the solutions, and the winner or winners will be announced and contacted before March 1, 1985. If several solutions are found, the prize will be shared amongst the solvers.