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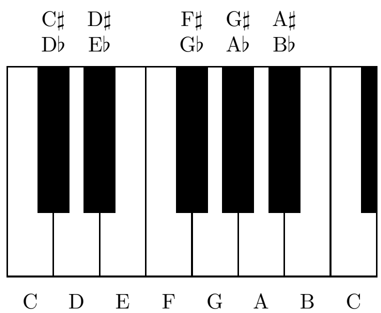
ARITHMETIC AND MUSIC IN TWELVE EASY STEPS*

First we must introduce some musical terminology, with apologies to readers who are already familiar with it. Consider a piano keyboard, part of which is shown below.



The white keys are labelled with the first seven letters of the alphabet: A, B, C, D, E, F, G. After using all of these we start again. (Note above that A follows G, and that C appears at both the left and right hand ends of the diagram.) A black key is given the name of the white key just below it, with a *sharp* (♯) added; or of the white key just above, with a *flat* (♭) added. Thus the leftmost black key in the diagram is called C♯ or D♭ (pronounced "C sharp", "D flat").

The interval from any key to the next is called a *semitone*. This is the smallest interval used in most Western music. For example, C – C♯, G♯ – A, B – C are all semitones. The interval from any key to the next key of the same name (for example C – C, E♭ – E♭) is called an *octave*. Counting five steps up a scale (including the first and last notes) gives the interval of a *fifth*, sometimes, for emphasis, called a *perfect fifth*. (Musical readers will know that there are other kinds of fifths, but we shall not be concerned with them in this article.) Instances of perfect fifths are C – G, B – F♯ and G♭ – D♭.

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By counting the keys in the figure you can easily find that an octave contains twelve semitones. In this article I attempt to answer the question, "Why twelve?" Why are there not, say, eleven, thirteen or forty one semitones to the octave? I will try to show that there are very good reasons for having twelve notes in an octave.

I should begin by pointing out that this article is not a historical one. I am not suggesting that people knew the following theory and therefore chose twelve semitones to the octave. Rather, I think that the theory in some sense constitutes a law of nature, and that given certain musical requirements, the appearance of an octave containing twelve notes is almost forced – just as you need not know anything about energy or gravity in order to hurt yourself if you fall downstairs!

Musical sounds are caused by regular vibrations in the air. The number of vibrations per second causing any particular note is the *frequency* of that note in units of *Hertz* (Hz). For example, the modern standard of orchestral pitch is established by *defining* the note A above middle C to have a frequency of exactly 440 Hz – that is, 440 vibrations per second. It is found by observation (and backed up by psychological and physiological theories) that when two notes of different pitches are sounded simultaneously or consecutively, the result is most pleasing to the ear if the *ratio* of the frequencies of the pitches is a simple fraction. The simplest possible fractions are $\frac{2}{1}$ and $\frac{3}{2}$, and these correspond to the intervals of the octave and the perfect fifth respectively. Middle C, for example, has a frequency of 262 Hz; a perfect fifth above is G with a frequency of 393 Hz; the octave above middle C is the C with frequency 524 Hz. The octave and fifth, being the "simplest" intervals, are fundamental to the music of virtually all cultures.

Now imagine that we start off with a single note and wish to build intervals on top of it in order to create a system of notes which we can use for melodies (and possibly harmonies too). In view of what has just been said, the best intervals to use will be octaves and perfect fifths. So we can take our starting note and add the note an octave above, then the note an octave above that, and so on. Returning to the initial note we construct a second series consisting of the note a fifth higher, the note a fifth higher again, and so on. The crucial requirement now is that *these two series should coincide* at some point: for then we will have a coherent system of individual notes rather than an infinite mess.

Time to turn to mathematics! According to the previous paragraph, we want an exact number of fifths (say p) to equal an exact number of octaves (say q). Now since the frequency ratio of a fifth is $\frac{3}{2}$, the ratio of p fifths piled up on top of each other is $\left(\frac{3}{2}\right)^p$; likewise q octaves give a ratio of 2^q. So we want

$$\left(\frac{3}{2}\right)^p = 2^q \tag{1}$$

for some positive integers p, q. This is the same as

$$\frac{3^p}{2^p} = 2^q \ ,$$

or

$$3^p = 2^{p+q} \ ;$$

but this is impossible as the left hand side is odd and the right hand side is even. So our two series will never coincide.

Thus the task we have set ourselves is impossible. The best that we can do is to find p, q such that (1) is very nearly true. To this end we take the logarithm of each side,

$$p \log \tfrac{3}{2} = q \log 2 \ ,$$

and rearrange to yield

$$\frac{p}{q} = \frac{\log 2}{\log \frac{3}{2}} \ . \tag{2}$$

We know that this equation has no solution, and we want to find good approximate solutions.

The problem of finding good fractional approximations to a given number may generally be attacked by the use of *continued fractions*. (This technique is most useful when, as in the present case, the number cannot be represented *exactly* by a fraction; however the same ideas can be used to find a fraction with small denominator which is close to

a "complicated" fraction such as $\frac{1234567}{2345678}$.) A continued fraction is an expression such as the following:

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} \ ,$$

where each fraction line extends over the whole of the subsequent part of the expression. The numbers (in this example) $3, 7, 15, 1$ are called the *partial quotients* of the continued fraction. We can evaluate such an expression in the obvious way "from the bottom up":

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = 3 + \cfrac{1}{7 + \cfrac{1}{16}} = 3 + \frac{16}{113} = \frac{355}{113} \ .$$

However there is also a "top down" method, which apart from being slightly faster, gives much more information about the continued fraction (as we shall see). First construct a table

		3	7	15	1
0	1				
1	0				

in which the left hand elements are always $0, 1, 1, 0$ as shown, and the top row consists of the partial quotients of the continued fraction. Now complete the table: the number to be written in any space equals the partial quotient above it in the top row, times the number on the left of the space, plus the next number to the left again. Thus to calculate the second row we have

$$3 \times 1 + 0 = 3 \ , \ \ 7 \times 3 + 1 = 22 \ , \ \ 15 \times 22 + 3 = 333 \ , \ \ 1 \times 333 + 22 = 355 \ .$$

Filling in the third row similarly we obtain the complete table

		3	7	15	1
0	1	3	22	333	355
1	0	1	7	106	113

The answer, $\frac{355}{113}$, appears as if from thin air at the end of the second and third rows! (By looking carefully at the table you may be able to guess what familiar number this particular example is related to.) In general, a continued fraction

$$a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_n}}$$

can be evaluated by completing the table

		a_0	a_1	$\cdots$	a_n
0	1				
1	0				

to yield

		a_0	a_1	$\cdots$	a_n
0	1	p_0	p_1	$\cdots$	p_n
1	0	q_0	q_1	$\cdots$	q_n

where

$$p_k = a_k p_{k-1} + p_{k-2} \ , \quad q_k = a_k q_{k-1} + q_{k-2}$$

and to start things off

$$p_{-2} = 0 \ , \ p_{-1} = 1 \ , \qquad q_{-2} = 1 \ , \ q_{-1} = 0 \ .$$

Once the table is completed, the value of the continued fraction can be read off from the end of the second and third rows:

$$a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_n}} = \frac{p_n}{q_n} \ .$$

The intermediate fractions

$$\frac{p_0}{q_0} \ , \ \frac{p_1}{q_1} \ , \ldots , \ \frac{p_{n-1}}{q_{n-1}}$$

formed from the second and third rows of the table, and also p_n/q_n itself, are called the *convergents* of the continued fraction. Thus the convergents of $\frac{355}{113}$ are $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$ and $\frac{355}{113}$ itself. The importance of continued fractions and convergents lies in the following fact: *the "best" fractional approximations* (in a certain sense) *to a given number are its convergents*. Now let's return to our musical problem and use this principle to find fractions which make (2), and therefore (1), true to a good approximation.

So far we have only found the convergents of a known continued fraction. Here we want to find the convergents of the number $\log 2/\log \frac{3}{2}$, so we must first calculate its continued fraction. The procedure is a little tricky in this instance, so let's do a simpler example first.

Suppose that we wish to write

$$\frac{37}{13} = a_0 + \frac{1}{a_1 + \cdots} \ .$$

Now a_1 will be a positive integer, therefore at least 1; so $1/(a_1 + \cdots)$ will be less than 1. This means that $\frac{37}{13}$ is equal to an integer a_0 plus something less than 1: thus a_0 must be the *largest integer not exceeding* $\frac{37}{13}$. This is called the *integer part* of $\frac{37}{13}$, and is denoted by square brackets. Thus we find

$$a_0 = \left[\frac{37}{13}\right] = 2 \ ,$$

because 13 goes into 37 twice, but not three times. Hence

$$\frac{37}{13} = 2 + \frac{11}{13} = 2 + \frac{1}{13/11} \ .$$

Now we apply the same process to the remainder $\frac{13}{11}$. We find $\left[\frac{13}{11}\right] = 1$, so

$$\frac{37}{13} = 2 + \cfrac{1}{1 + \cfrac{2}{11}} \quad = 2 + \cfrac{1}{1 + \cfrac{1}{11/2}} = 2 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{2}}} \ .$$

We have found the continued fraction for $\frac{37}{13}$. In summary, the continued fraction for any number is found by (i) extracting the integer part; (ii) writing the remainder as a fraction with numerator 1; and (iii) repeating (i) and (ii) as often as necessary. Note that if a continued fraction stops after a finite number of terms the result will be a fraction, as we saw above. But we know that $\log 2/\log \frac{3}{2}$ cannot be written as a fraction! Therefore its continued fraction expansion will never stop: it will have infinitely many terms. (It's rather like an infinite decimal.) This will not worry us – we'll just calculate enough terms to find a convergent which is accurate enough for our needs.

On to the continued fraction for $\log 2/\log \frac{3}{2}$. The first difficulty is to find $\left[\log 2/\log \frac{3}{2}\right]$. This could be done by using a calculator to work out $\log 2/\log \frac{3}{2}$ and then just dropping the part after the decimal point. However a neater method is as follows. Let n be a positive integer. Then

$$n \le \frac{\log 2}{\log \frac{3}{2}} \quad \Leftrightarrow \quad n \log \tfrac{3}{2} \le \log 2 \quad \Leftrightarrow \quad \left(\tfrac{3}{2}\right)^n \le 2 \quad \Leftrightarrow \quad 3^n \le 2^{n+1} \ .$$

Now $\left[\log 2/\log \frac{3}{2}\right]$ is the largest integer n for which $n \le \log 2/\log \frac{3}{2}$; that is, the largest integer for which $3^n \le 2^{n+1}$. Here the only choice is $n = 1$. So we begin

$$\frac{\log 2}{\log \frac{3}{2}} = 1 + \frac{\log 2 - \log \frac{3}{2}}{\log \frac{3}{2}} = 1 + \frac{\log \frac{4}{3}}{\log \frac{3}{2}} = 1 + \frac{1}{\log \frac{3}{2}/\log \frac{4}{3}} \ . \qquad (3)$$

For the next step we calculate

$$n \le \frac{\log \frac{3}{2}}{\log \frac{4}{3}} \quad \Leftrightarrow \quad n \log \tfrac{4}{3} \le \log \tfrac{3}{2}$$
$$\Leftrightarrow \quad \left(\tfrac{4}{3}\right)^n \le \tfrac{3}{2} \quad \Leftrightarrow \quad 2 \times 4^n \le 3^{n+1} \ .$$

Again this is true only for $n = 1$. Thus

$$\frac{\log \frac{3}{2}}{\log \frac{4}{3}} = 1 + \frac{\log \frac{3}{2} - \log \frac{4}{3}}{\log \frac{4}{3}} = 1 + \frac{\log \frac{9}{8}}{\log \frac{4}{3}} = 1 + \frac{1}{\log \frac{4}{3}/\log \frac{9}{8}} \ ,$$

and substituting into (3) gives

$$\frac{\log 2}{\log \frac{3}{2}} = 1 + \cfrac{1}{1 + \cfrac{1}{\log \frac{4}{3}/\log \frac{9}{8}}} \ .$$

Repeating the process,

$$n \le \frac{\log \frac{4}{3}}{\log \frac{9}{8}} \quad \Leftrightarrow \quad n \log \tfrac{9}{8} \le \log \tfrac{4}{3}$$
$$\Leftrightarrow \quad \left(\tfrac{9}{8}\right)^n \le \tfrac{4}{3} \quad \Leftrightarrow \quad 3 \times 9^n \le 4 \times 8^n \ .$$

Now this is true for $n = 1$ and 2; we require the largest possible value of n, that is, $n = 2$. As before,

$$\frac{\log \frac{4}{3}}{\log \frac{9}{8}} = 2 + \frac{\log \frac{4}{3} - 2\log \frac{9}{8}}{\log \frac{9}{8}} = 2 + \frac{\log \frac{256}{243}}{\log \frac{9}{8}} = 2 + \frac{1}{\log \frac{9}{8}/\log \frac{256}{243}} \ .$$

I'll leave you to work out a few more terms if you wish. You can see that the numbers involved grow rapidly and so the calculations become more and more time-consuming. I wrote up this procedure on a powerful computer equipped with special software for dealing with large numbers, and it took the computer 30 seconds* to calculate only the first seven terms! The answer this far is

$$\frac{\log 2}{\log \frac{3}{2}} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cdots}}}}}} \ .$$

Once we have the partial quotients it is a much easier job to find the table of convergents. It begins

		1	1	2	2	3	1	5	···
0	1	1	2	5	12	41	53	306	···
1	0	1	1	3	7	24	31	179	···

* This was written in 1989. Nowadays the calculation would be virtually instantaneous!

Now we can interpret our results in terms of the musical problem. Recall that the convergents give us approximate solutions p/q to (2), and this says (compare (1)) that

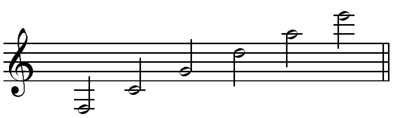
$$\left(\frac{3}{2}\right)^p = 2^q$$

approximately; and this means that p perfect fifths approximately equal q octaves.

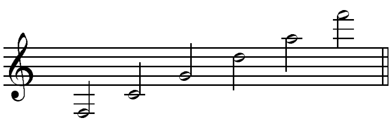
Our first convergent is $\frac{1}{1}$. Thus as a first approximation $\frac{3}{2} = 2$, or one fifth equals one octave. These are obviously terrible approximations! Never mind – it gets better further on.

The second convergent is $\frac{2}{1}$, which means that two fifths approximately equal one octave, or that C – D is approximately the same as C – C. This is better than the previous attempt, but still much too crude to be of any practical use.

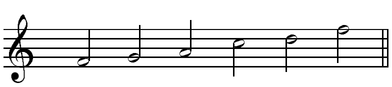
The third approximation is that five fifths equal three octaves. If we start with an F (for no reason except to fit all the notes on one staff), five fifths look like this:



and indeed the top E is just short of three octaves above the bottom F. If we replace the E by an F we obtain the series of notes F – C – G – D – A – F:



Reordering these notes and transposing them so that they lie within one octave we have



the pentatonic (five-note) scale F – G – A – C – D – F which is commonly found in folk music (particularly in Chinese and Japanese, Scottish and Irish, and African music). Raising the whole scale by a semitone gives the black keys of the piano. If you play a tune using only the black keys you should find that it has a "folk music" flavour. Thus we see that the approximation $\log 2/\log \frac{3}{2} = \frac{5}{3}$ leads to satisfactory musical results.

The next approximation is $\frac{12}{7}$. If we start on a (very!) low F and build twelve fifths on top of it we obtain the series

Here the top E♯ is approximately the same as the F seven octaves higher than our starting note. As above we can rearrange the notes within the compass of one octave and replace the E♯ by F

to obtain the usual chromatic scale, which corresponds to playing all the black and white keys of the piano from F to F.

Musical readers will probably have said by now, "But E♯ equals F exactly. There is no approximation necessary." Not so! Indeed E♯ and F are the same note on a piano, but this is only because the tuning of a piano already incorporates the approximation we are discussing. Acoustically, the frequency ratio for the interval F – E♯ from the first to the last note of the series at the top of page 10 is $\left(\frac{3}{2}\right)^{12}$, whereas the ratio from the bottom F to the top F which approximates E♯ is 2^7. These ratios are not equal; therefore E♯ does not equal F. In fact none of the intervals on a piano is correctly in tune, except for the octaves. This is because instead of concentrating the whole error into one interval (for example the interval A♯ – F, which as above ought to be A♯ – E♯), the error is spread equally over all twelve intervals. Thus the frequency ratio of a fifth in this system (known as "equal temperament") is the number r such that

$$r^{12} = 2^7 \ ;$$

that is, $r = \sqrt[12]{128} = 1.4984$, which is not quite $\frac{3}{2}$.

Our study of continued fractions would appear to give a satisfactory answer to the question, "Why are there twelve semitones in an octave?" We seek a system of notes which will accommodate both fifths and octaves to a high degree of precision (for no system can do so exactly), and are thus led to a problem of approximation which we solve by continued fractions. This gives a usable system of five fifths to three octaves; however, this system is usually considered too limited to provide enough interesting musical possibilities, so we move on to the next approximation, which gives us the (nowadays) standard chromatic scale of twelve notes to the octave.

There is no theoretical reason why we should not use even better approximations: the next would result in a 41–note scale. However with such fine divisions of notes the practical problems in performance of the music become severe. Of course this only applies to human performers – there is probably no reason why computer music should not be written with systems of 41 notes, 53, 306 or even more. (Actually the American composer Harry Partch has written music for human performance using a 43 note scale. Clearly his system is not based on the principle of building up fifths and octaves which we have considered, and unfortunately I have been unable to find out his reasons for choosing 43 notes to the octave.)

There are many other interesting problems of approximation which can be solved using continued fractions, and which I may write about in a future issue of *Parabola*.

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