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ORDER, RANDOMNESS AND CHAOS

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A *time series* is a sequence of values $x_1, x_2, x_3, \ldots$, usually representing measurements of some quantity at equal intervals of time. For example, x_n might denote the annual rainfall in Sydney in the n-th year of records, or the exchange rate for the Australian dollar on day n, or the number of koalas found in a particular area in the n-th month. Below are plotted the time series for three systems which I have artificially generated.

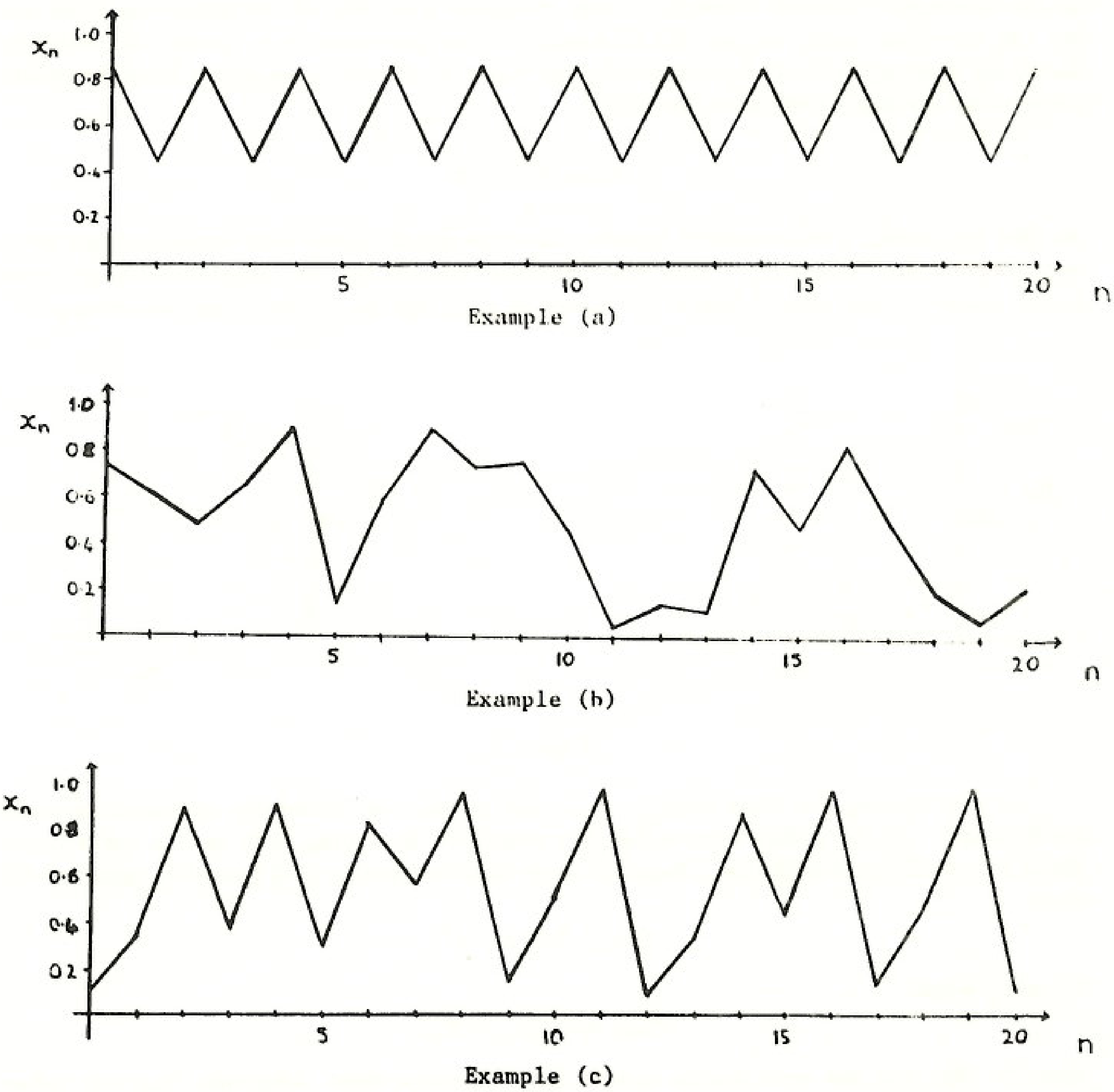

Figure 1.

Example (a) clearly has a simple pattern to it, and we would feel quite confident in predicting the future values of this time series. The other two however, seem to obey no simple rule. Such random looking time series are common in real world examples, but until recently it has been thought that there was little hope in being able to predict the future behaviour of such systems. And indeed often such prediction is impossible. Many a gambler would like to be able to make predictions about future rolls of a roulette wheel, but on a fair wheel all one can do is give probabilistic predictions of the relative frequencies of the various numbers appearing; there is no way of telling what the next number will be. The data for Example (b) was derived from taking the last two digits from a page in the Sydney telephone directory - an essentially random process - and unless I tell you where in the directory I started, there is no way of telling what any of the future numbers will be. On the other hand the data for Example (c) was generated by a simple rule, whereby the value at every point is determined by the value at the previous point. In other words, there is a function f such that, for all n,

$$x_{n+1} = f(x_n).$$

At first this seems a little hard to believe, since the graph of the time series appears to be very complicated. However, to show the difference between the two "random looking" times series above, I have plotted below the values of x_{n+1} against x_n for these two examples.

Figure 2.

As you can see, for Example (b), x_{n+1} and x_n seem to be totally unrelated, whereas for Example (c), the points seem to lie on a parabola. By trying to fit such a curve to the data, you would probably find a function very close to the one that I used to generate the data:

$$f(x) = 3.9(x - x^2).$$

In other words

$$x_{n+1} = 3.9(x_n - x_n^2).$$

Systems like this one give simple models of population sizes. Although they are rather too unsophisticated to be accurate models, they do share the basic properties of many populations that they grow very quickly when their size is small, and die off through overcrowding if the population becomes too large.

Now that we know the function which generates the data for Example (e), it should be a simple task to plug in the last point in the time series and then perform the above iteration to forecast the values for as far into the future as we wish. Unfortunately things are not so simple. Let's start at the first point on the time series and see if we can predict the values that appear in the graph. The first value x_0, appears to be about 0.1. Plugging that into the iteration gives $x_1 = 3.9(0.1 - 0.01) = 0.351$. Repeating this gives the time series shown below. As you can see, it agrees pretty well with the original Example (e) for the first 5 or so points, but soon the two time series are behaving very differently. So what went wrong? The problem is that the system that we are studying is *chaotic*.

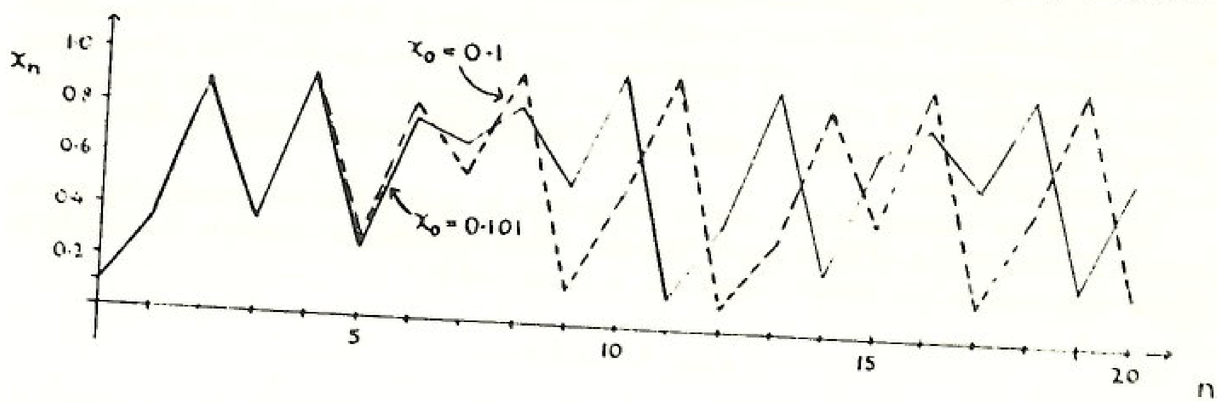

Figure 3.

Chaos Theory is a new and exciting branch of mathematics which has many links to other areas of research. Recently, the University of New South Wales hosted a large international conference on Chaos Theory which was attended by many of the major researchers in the field, and some of you may have been lucky enough to attend the public lectures given by Professors Benoit Mandelbrot and Robert May. Indeed anyone who attended Professor May's talk will recognise much of the mathematics in this paper. So what do we mean whan we say that a system is chaotic? Chaotic systems are characterised by two important properties:

(i) The system is governed by a simple set of rules;

(ii) The behaviour of the system is highly sensitive to the initial conditions of the system.

It is this sensitivity to initial conditions that is causing the problems shown in Figure (3). When I generated the original example I started off with initial value $x_0 = 0.101$. Whilst this is only different from the value we used the second time by 1 part in 100, the two time series soon diverge. Even if we had been very, very accurate in reading the value of x_0, the two paths would have parted quite rapidly - unless we had chosen the exact same value. And even if we did start with exactly the same value, if we had performed the calculations on a different computer, different rounding errors would eventually have caused the time series calculated to be quite different from the original one.

In practice it is usually impossible to know exactly the value of a certain quantity, so one might be tempted to ask if we have gained anything from discovering the ***dynamics*** (the rules) of the system. As the above graph shows, what we have gained is the ability to give short term predictions. Thus we might regard Example (c) as lying halfway between Example (a), which was very regular, and Example (b) which was random. In our chaotic system, once we know the rule, and (approximately) the value of x_n, we can at least accurately predict the next few values.

This has had many important implications in other areas of science. Previously it had been thought that if one knew the equations that controlled the weather and one knew the present state of the earth's atmosphere accurately enough, then, given a big enough computer you could make accurate long term weather predictions. We now know however that the equations which govern the earth's atmospheric system are chaotic and so that there is a theoretical limit to how far ahead such predictions can be made. Even if we fill the sky with weather satellites and employed all our fastest computers, it is impossible to predict the weather for more that about three weeks ahead. Eventually even the tiniest error will be magnified to an extent which makes any prediction useless. This phenomenon is often given the rather colourful name of the ***butterfly effect***, because the disturbance caused by a butterfly flapping its wings in say, Tokyo, will eventually have an effect on our weather here in Sydney.

Much of the research in this area is being done in finding ways of recognising which systems are chaotic, or at least partly chaotic. Many systems, for example, contain both chaotic and random (or "noise") elements. In other words,

$$x_{n+1} = f(x_n) + \varepsilon_n$$

where ε_n is some random noise component in the reading. Even if the noise component is quite large, it is often very useful to be able to "explain" at least some of the behaviour of the time series. Other work is being done on ways of predicting the future of such systems without explicitly finding the function f.

Being able to tell that a system is chaotic is not always easy at first glance. Example (a) above was generated by the rule

$$x_{n+1} = 3.42(x_n - x_n^2)$$

which is very similar to the rule which generated Example (c). Those of you with computers (or even programmable calculators) might like to experiment with rules determined by the functions

$$f_a(x) = a(x - x^2)$$

for different values of a and with different starting points x_0. Iteration procedures like this are easy to program in most computer languages, and computers are very good at tasks such as this which involve a large number of repetitions of a simple step. Indeed because of this, the introduction of computers was a major force in the stimulation of areas such as Chaos Theory which involve iteration.

The interesting behaviour occurs when a is between 3 and 4. For some values the time series tends towards a constant. Then as a increases, the system becomes periodic of period 2 (like Example (a)), then period 4, then 8, ... and finally becomes chaotic. A little experimentation will enable you to discover the values of a at which this "period doubling" takes place. Sometimes this behaviour takes a very long time to settle down. For example, if $a = 3$, the system converges towards 2/3, but unless we start very close to the limit value, it will take a very large number of iterations before every x_n is within say 0.0001 of the limit. You might also try some experiments with different types of functions determining the dynamics of the system. Just pick a function which maps the unit interval [0, 1] onto itself and start iterating. Does a pattern form, or is the system chaotic? One function you might try is the tent map

$$f(x) = 2\min\{x, 1 - x\}.$$

Chaos Theory challenges the idea which has been prevalent in the scientific community for generations that a simple set of rules generates a simple outcome. Many such systems have been neglected from science and mathematics courses because they seemed to be too hard to understand or because they did not have elegant solutions. Yet hidden within the complexities of these systems there is much elegance and beauty, and many important applications. Anyone interested might try to find out about "fractals", such as the beautiful Mandelbrot and Julia sets. That however is another story, one which you might chase up in the references below.

References

1. James Gleick, *Chaos: Making a New Science*, (New York: Viking Press 1987).

2. Benoit Mandelbrot, *The Fractal Nature of Geometry*, (San Fransisco: W.H. Freeman 1982).

3. Heinz-Otto Peiten and Peter H. Richter, *The Beauty of Fractals*, (New York: Springer 1986).