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VARIATIONS ON A TRIGONOMETRIC THEME

By

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1. Introduction

Here in a problem of the sort that could possibly be set in a 4-unit paper:

> "Let A, B, C be nonnegative angles such that $A+B+C = 180°$.
> Show that $4\sin A\sin B\sin C = \sin 2A + \sin 2B + \sin 2C$ and that the maximum of the expression is $3\sqrt{3}/2$."

In fact this piece of mathematics can be done (and better understood) with the use of a little circle geometry which is well within the scope of 2-unit material. Before we investigate that, let's do it as an exercise in double angle trigonometric formulae and differential calculus the sophisticated but silly way!

2. Adult toys

Because the angles add to make a straight angle, we know that

$$\sin C = \sin(A+B)$$

and therefore

$$\begin{aligned}4\sin A\sin B\sin C &= 4\sin A\sin B\sin(A+B)\\ &= 4\sin A\sin B(\sin A\cos B + \cos A\sin B)\\ &= 2\sin^2 A.2\sin B\cos B + 2\sin^2 B.2\sin A\cos A\\ &= 2\sin^2 A\sin 2B + 2\sin^2 B\sin 2A\\ &= \sin 2A + \sin 2B + (2\sin^2 B - 1)\sin 2A + (2\sin^2 A - 1)\sin 2B.\end{aligned}$$

Since $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$, we have now proved that

$$4\sin A\sin B\sin C = \sin 2A + \sin 2B - \sin 2A\cos 2B - \sin 2B\cos 2A.$$

Of course we recognize

$$\sin 2A\cos 2B + \sin 2B\cos 2A$$

as an expanded version of

$$\sin(2A + 2B)$$

which, in turn, equals

$$-\sin 2C$$

because $2A, 2B, 2C$ and to give $360°$.

This completes the proof that, for $A + B + C = 180°$,

$$4 \sin A \sin B \sin C = \sin 2A + \sin 2B + \sin 2C.$$

In finding the maximum we can work with either the left hand side or the right hand side, whichever we prefer. It is usually easier to work with a sum so let's be perverse and consider

$$F(A, B) = 4 \sin A \sin B \sin(A + B).$$

In order to use the standard tricks of calculus, we should switch to radians and consider B fixed. Taking the derivative of F with respect to A we find

$$4 \cos A \sin B \sin(A + B) + 4 \sin A \sin B \cos(A + B)$$

. For a critical point this should be zero and since we are seeking a maximum we may as well suppose that $\sin B \neq 0$. Accordingly we solve

$$\cos A \sin(A + B) + \sin A \cos(A + B) = 0;$$

in other words

$$\sin(2A + B) = 0.$$

Our standing assumption is that $A + B + C = \pi$ and A, B, C are nonnegative, so this gives

$$2A + B = \pi$$

when we rule out the case $A = B = 0$.

We can save some effort by some subtle reasoning at this point. Because $F(A, B)$ is symmetric in A, B, the foregoing shows also that, for fixed A, the maximum occurs when $2B + A = \pi$. The maximum of $F(A, B)$ must occur therefore when both

$$2A + B = \pi, \quad 2B + A = \pi.$$

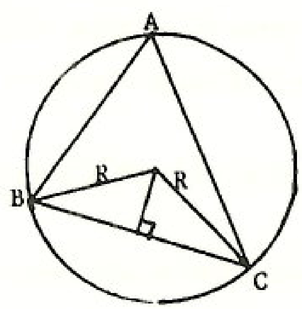
Naturally this forces $A = B = \pi/3$ and confirms that the maximum of $F(A, B)$ is $3\sqrt{3}/2$.

3. Childish things

Let's start afresh from a more primitive viewpoint. The angles in question form the angles of some triangle ABC and, like every triangle, this triangle determines a circle passing through its vertices. Let us denote the radius of this circle by R and note that

$$a = 2R \sin A,$$

where a is the length of the side BC which lies opposite A. The following picture should jog your memory:



We know also that the area, Δ, of triangle ABC equals $\frac{1}{2}ab \sin C$, and thus we find

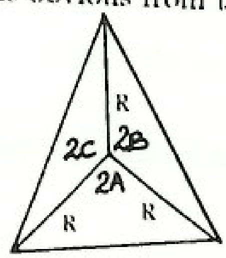
$$\begin{aligned}\Delta &= \frac{1}{2}(2R \sin A)(2R \sin B) \sin C \\ &= 2R^2 \sin A \sin B \sin C.\end{aligned}$$

We are concerned with maximising $2\Delta/R^2$. In fact now we realize that the problem of maximising $4 \sin a \sin b \sin C$ subject to $A + B + C = 180°$ is just the problem of finding the shape of a triangle which has the greatest area given its circumcircle. Fix the circle and consider a triangle of maximum area. Consider any side, BC say, and draw its perpendicular bisector which is, of course, a diameter. The longest perpendicular distance from BC to a point on the circle must be attained along that diameter. Therefore A lies on the diameter (because ABC has maximum area) and it follows that $AB = AC$. We could have started with any other side, so this proves that ABC is equilateral and gives the maximum as $3\sqrt{3}/2$.

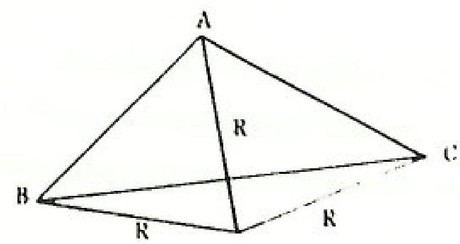
For completion we also reprove our original trigonometric identity. In view of our formula for Δ, this amounts to showing that

$$\Delta = \frac{1}{2}R^2(\sin 2A + \sin 2B + \sin 2C).$$

In the acute-angled case this is obvious from the following diagram:



In general we may need to subtract an area:



Here Δ is made up by adding the areas of AOB and AOC and subtracting that of BOC. The subtraction corresponds to the fact that $\sin 2A$ is negative. This completes the simple way of solving the problem.

4. Conclusion

It seems to me that a mark of good science is its explanatory power and that a mathematician should always be trying to be as simple minded as possible. On the other hand there is nothing wrong with being a bit fascinated by mathematical technology in its own right. A high-school student should enjoy the manipulations of section 2 for their own sake and should get a charge out of being able to master such techniques. Ideally though, there should also be time to mess around and ask "what's really going on?" For my taste, section 3 comes nearer to answering that.