Parabola Volume 26, Issue 1 (1990)

THE GEOMETRY OF COLOUR

David Tacon

In a recent article of **Parabola** (Vol.25 No.3) Esther Szekeres showed how we could prove many geometrical results by thinking in terms of centres of mass. An analogous idea is to think in terms of colour. If red paint and yellow paint are mixed in some proportion a shade of orange is produced. Let us write

$$C = rR + yY$$

to indicate that the colour C is produced by adding r litres of red paint to y litres of yellow paint. We assume

$$r + y = 1$$

so that 1 litre of C is produced.

We can further mix any two shades of orange to produce another shade of orange. The final mixture can be easily computed in terms of R and Y as follows:

$$\text{if} \quad C_1 = r_1 R + y_1 Y \quad \text{and} \quad C_2 = r_2 R + y_2 Y$$

then

$$aC_1 + bC_2 = (ar_1 + br_2)\ R + (ay_1 + by_2)Y$$

Clearly this is *physically* true. Note that if $r_1 + y_1 = 1$, $r_2 + y_2 = 1$ and $a + b = 1$ then the final mixture also contains 1 litre since

$$\begin{aligned} ar_1 + br_2 + ay_1 + by_2 &= a(r_1 + y_1) + b(r_2 + y_2) \\ &= a + b = 1. \end{aligned}$$

The key idea is that we can now display all of these colours *geometrically*. Choose any line segment and paint it so that the colour of the segment varies continuously from red through all shades of orange to yellow.

It is reasonable to paint P the colour $C = rR + yY$ where $r = PY/RY$, $y = RP/RY$ (so that $r + y = 1$). Note that the *amount*, r, is proportional to the distance of P from Y, which ensures that Y is painted yellow and R red.

This procedure can be used to paint a line segment once the colours of the endpoints are known. We shall say that a line segment is *painted linearly* if this procedure has been used to paint it. It is perhaps obvious that a line which is painted linearly between its endpoints is also painted linearly between any two of its points. This is essentially the content of our basic result which we soon prove. For definiteness let's understand that if P and Q are points, and if $a + b = 1 (a, b > 0)$ we define $X = aP + bQ$ as the unique point X on the segment PQ such that

$$\frac{PX}{PQ} = b \quad \left(\text{and } \frac{XQ}{PQ} = a\right).$$

Basic Result. If

$$P_1 = p_1P + q_1Q \quad \text{and} \quad P_2 = p_2P + q_2Q$$

then

$$aP_1 + bP_2 = (ap_1 + bP_2)P + (aq_1 + bq_2)Q$$

Before proving the result its important to appreciate what it is saying. It is saying that the point represented by the left-hand side of the equation is the same point as that represented by the right-hand side of the equation. Once proven it permits the usual algebraic manipulations to express a mixture of mixtures of P and Q as a mixture of P and Q.

Proof.

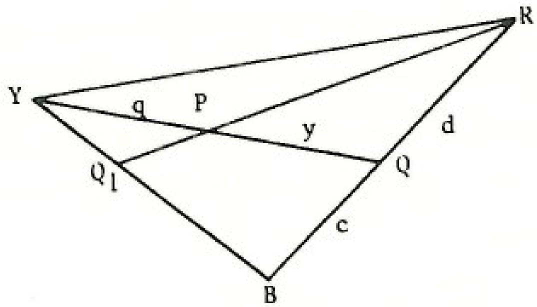
Let T be the point $aP_1 + bP_2$. Then

$$\begin{aligned} PT &= PP_1 + P_1T = q_1PQ + bP_1P_2 \\ &= q_1PQ + b(PQ - PP_1 - P_2Q) \\ &= q_1PQ + b(PQ - PP_1 - P_2Q) \\ &= PQ[q_1(1 - b) + b(1 - p_2)] \\ &= PQ(q_1a + bq_2) \quad \text{as we require.} \end{aligned}$$

In a similar way, we obtain

$$TQ = (ap_1 + bp_2)PQ. \qquad \blacksquare$$

These ideas naturally suggest how we might paint a triangle using 3 colours, say red, yellow and blue.



The boundary is easily coloured since each point requires only two colours (corresponding to the colours of the two end vertices). To colour the point P extend YP to Q. Then mix R and B in the proportions c to d to obtain the colour of Q. We can now combine q parts of this mixture with y parts of yellow to obtain the colour of P. We say P *uses y parts of yellow.* Of course we could have just as easily have obtained a green colour for Q_1 and combined it suitably with, say, r parts of red.

If we similarly define the number of parts b of blue used by P we are lead to write

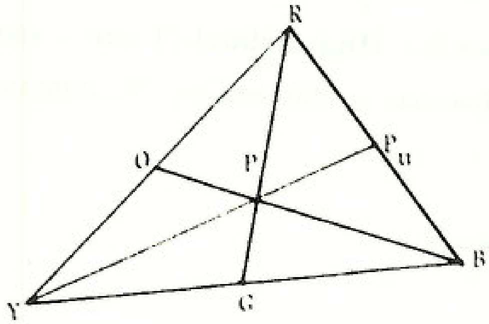
$$P = bB + rR + yY \qquad (*)$$

It is not clear that $b + r + y = 1$. Nor is it clear that the same colour is obtained by this mix as would have been obtained in our initial mixture of y parts of Y with $q = 1 - y$ parts of Q. Nevertheless on reflection it is what we might expect. Further, if $b + r + y = 1$ and $b, r, y \geq 0$, there is precisely one point P in the triangle for which (*) holds and, moreover, it is the case that any segment in the triangle is painted linearly by this procedure. Let us assume these things to simplify our discussion.

What sort of results can we hope to prove with these ideas? We would expect to be able to use them in any problem which concerns points on a line and the ratio of distances between them. Here are the proofs of two theorems. The first is in Euclid and has a natural proof in terms of centres of masses as Esther explained.

Theorem. *The medians of a triangle meet in a point.*

Proof (via colours): Mix equal parts of R, B, Y to colour a unique point P as in the diagram.

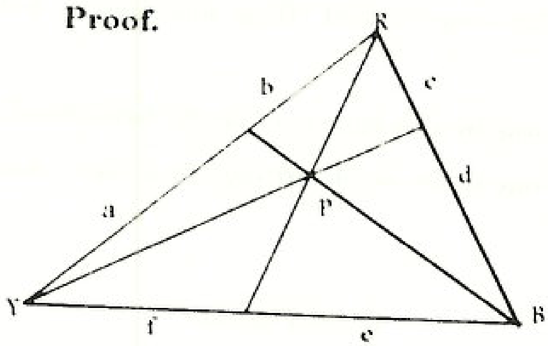


The colour of P may be obtained by first obtaining a pure orange $(\frac{1}{2}R + \frac{1}{2}Y)$ and then taking two parts of this orange and mixing it with one part blue. Two other similar methods give the same colour. Geometrically this one colour is $\frac{2}{3}$ of the distance from any vertex along the median. ■

There are relatively few important results of "Euclidean geometry" which are not in Euclid. The following is one of these results and was published by Giovanni Ceva in 1678. (Giovanni Ceva seems to have been a fairly minor Italian mathematician for he rarely gets more than a mention in books on the history of mathematics.)

Theorem. (Ceva) *If lines through P are drawn from each vertex of a triangle forming the lengths a, b, c, d, e and f as in the diagram than $ace = bdf$*

Proof.



Suppose $P = bB + rR + yY$. By hypothesis P_1 uses B and R in the proportion c to d. P mixes P_1 with Y and hence also uses B and R in the same proportion. Therefore

$$\frac{c}{d} = \frac{b}{r}$$

Similarly,

$$\frac{e}{f} = \frac{y}{b} \quad \text{and} \quad \frac{a}{b} = \frac{r}{y}$$

Multiplying these equations gives

$$\frac{ace}{bdf} = \frac{r}{y} \cdot \frac{b}{r} \cdot \frac{y}{b} = 1. \qquad ■$$

These two results on their own should be enough to convince us of the use of colour mixtures in solving problems in plane geometry. It is intuitively clear how to colour the points inside a tetrahedron in terms of the colours of its four vertices. With this intuition we should be able to reinterpret the results of Esther about such solids. Afficionados of the method are referred to the excellent article *The geometry of colour* by Melvin Hausner which appeared in **Enrichment Mathematics for High School** (Twenty-eighth year book) [published by The National Council of Teachers of Mathematics, Washington D.C., in 1963.]

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THE USE OF NAMES

"The name of the song is called '*Haddocks' Eyes.*"

"Oh, that's the name of the song, is it? Alice said, trying to feel interested.

"No, you don't understand," the Knight said, looking a little vexed.

"That's what the name is *called*. The name really *is*, '*The Aged Aged Man.*' "

"Then I ought to have said 'That's what the *song* is called'?" Alice corrected herself.

"No, you oughtn't: that's quite another thing! The *song* is called '*Ways and Means*': but that's only what its *called*, you know!"

"Well, what *is* the song, then?" said Alice, who was by this time completely bewildered.

"I was coming to that," the Knight said. "The song really *is* '*A- sitting On a Gate*': and the tune's my own invention."

Lewis Carroll

(or C.L. Dodgson,

an Oxford mathematician.)