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THE BANACH-TARSKI PARADOX AND THE SQUARING OF THE CIRCLE

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Mathematics is generally regarded as one of the few disciplines (some would say the **only** discipline) which is built on rock-solid foundations; once a theorem is proved, its truth cannot be challenged (unless of course the proof turns out to be faulty). Yet mathematics sometimes produces baffling paradoxes which wholly defy our intuition. One of the most remarkable of these paradoxes was discovered some 70 years ago by two Polish mathematicians, Stephan Banach and Alfred Tarski. They proved that two solid spheres, S_1 of radius 1 and S_2 of radius 2, can be partitioned into a **finite** number of subsets, namely $A_1, A_2, \cdots A_n$ and $B_1, B_2, \cdots, B_n$ respectively, so that A_1 is congruent to B_1, A_2 to $B_2, \cdots, A_n$ to B_n. By **partitioning** we mean that each point of S_1 is contained in exactly one A_i and each point of S_2 is contained in exactly one B_i – in symbols (if you prefer)

$$A_i \cap A_j = \phi \text{ for } i \neq j \quad \text{and } A_1 \cup A_2 \cup \cdots \cup A_n = S_1,$$

$$B_i \cap B_j = \phi \text{ for } i \neq j \quad \text{and } B_1 \cup B_2 \cup \cdots \cup B_n = S_2.$$

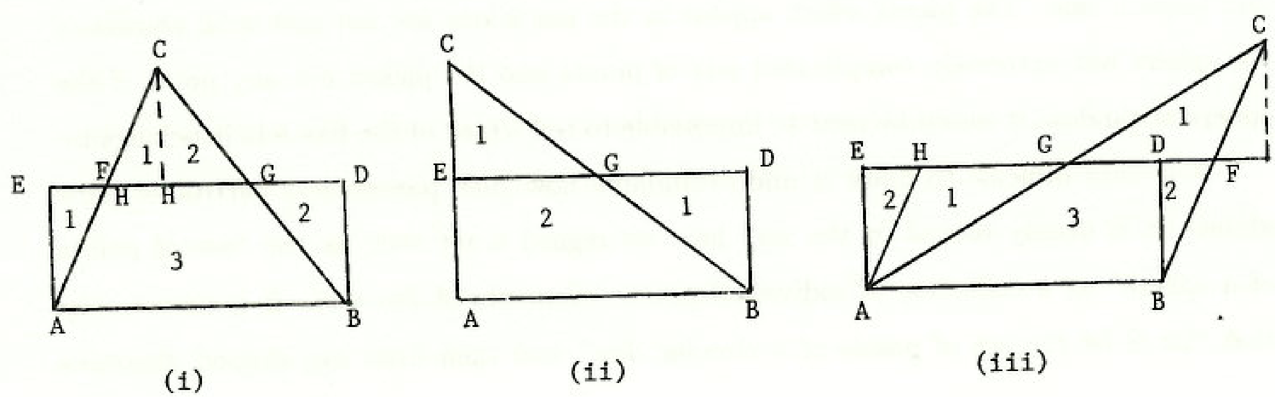
By congruent we mean: in the Euclidean sense, namely that for each i, A_i can be moved rigidly into B_i (without altering distances between the points of A_i).

Nothing could be further from our intuitive ideas of "truth" than the Banach – Tarski theorem. Consider the following version of the theorem, which is also true: A mathematical sphere of gold of radius 1 can be dissected into 5 pieces so that these 5 pieces can be reassembled to form two spheres of radius 1 each. An easy way to make a fortune. Yet neither Banach nor Tarski died particularly wealthy so there must be a catch somewhere, and there is one. The pieces which appear in the partitions are not just solid chunks of the sphere but extremely complicated sets of points and if I picked out any point of the sphere at random it would be next to impossible to tell which of the five sets it belongs to.

It is very difficult to make it understandable how such paradoxical partitions come about: it is deeply rooted in the way how we regard a set such as the "set of points of a sphere" as a collection of individual points (elements of the set). It is easy to say that "let S be the set of points of a circular disc" and then draw egg-shaped diagrams on a sheet of paper to illustrate properties of intersections, unions etc. But when it comes to visualizing sets as collections of individual elements our intuition can play tricks. Mathematical existence is not quite the same as "physical" existence, though perhaps not quite as different either; just think of the "existence" of elementary particles such as electrons, photons, neutrions etc.

Banach also showed that in the plane no such paradoxical situation arises: if two plane figures have congruent partitions (as we want to call for brevity the existence of finite pairwise congruent partitions of the two figures) then the figures must have the same area. This discovery of Banach prompted Tarski to pose the following question: do a circular disc and a solid square, both of area 1, have congruent partitions? This "squaring of the circle" has of course nothing to do with the famous problem of the ancient Greeks who asked whether one can construct, by means of a ruler and compasses alone, a square whose area is equal to the area of a circle with unit diameter.

Tarski's problem remained unsolved for almost 70 years – a long time for any mathematical problem. Before coming to the quite recent history of the problem, I want to remind you that the corresponding problem for triangles (and generally for polygons) has been solved almost 200 years ago. A theorem of Bolyai states that two triangles with equal area (by triangle here I mean together with its interior, a triangular region) can be dissected into pairwise congruent polygonal regions. This is not very difficult, though it requires some cleverness. First, it is easy to see that a triangle ABC with base AB, and a rectangle $ABDE$ with the same base but half the altitude can be dissected in pairwise congruent pieces. The following figures illustrate the cases when (i) both angles at A and B are acute, (ii) at A there is a right angle, and (iii) the angle at A is obtuse.



In all three figures F is the midpoint of AC, G is the midpoint of BC. In figure (i) $CH \perp ED$ and in figure (iii) $BH \parallel AF$. (Prove that in each figure the triangles marked by the same numbers are congruent). From here it is easy to deduce that two triangles with common base AB and equal altitudes have congruent dissections. (Proof: find a congruent dissection of both triangles and the common rectangle $ABCD$, and use the fact that if two figures T_1 and T_2 have congruent dissections with a third figure R then also T_1 and T_2 have congruent dissections with each other. This is not entirely obvious but I leave the proof to the reader). The general case of two triangles with equal areas can be settled similarly, using the previous fact. Details are again left to the reader.

Does it follow from Bolyai's theorem that two triangles with equal areas have congruent partitions in the sense of Tarski? Not quite. One doesn't quite know what to do with corner points in the dissections such as the point H in (i) which corresponds to two points namely E and D, in the rectangle. I want to show you how it is done, not so much for completeness sake but because it involves a very ingenious construction. Let me state the problem in a simplified form which shows up all the difficulties and also throws light on the types of sets which appear in the Tarski-type partition problems.

On the real number line take the closed interval [0,1], that is the set of numbers $0 \leq x \leq 1$, and the half-closed interval (0,1], that is the set of numbers $0 < x \leq 1$. Can one split up [0,1] and (0,1] into a **finite** number of subsets $A_1 A_2, \cdots, A_n$ and $B_1, B_2 \cdots, B_n$ respectively so that the set B_i for each $i = 1, 2, \cdots, n$ is obtained from A_i by a suitable translation, that is by adding to the members of A_i a fixed number c_i (positive, negative or 0)? If you try to find such sets you will probably get the impression that it cannot be done; the embarrassment is the point 0 in the interval [0,1] which doesn't seem to fit into any of the sets A_i. Surprisingly, it turns out that not only can the construction be carried out but actually three sets A_i suffice. Here is the construction; its understanding will no doubt require some concentration on your part.

Take any fixed **irrational** number between 0 and 1, say $\alpha = \sqrt{2} - 1 = 0,414213562 \cdots$. We now form the sequence of numbers

$$p_k = k\alpha - [k\alpha], \ k = 0, 1, 2, \cdots$$

where $[k\alpha]$, as usual, denotes the integer part of $k\alpha$ (that is, the largest integer less or equal to $k\alpha$). The first few terms of the sequence are $p_0 = 0$, $p_1 = \alpha = 0.414213562\ldots$, $p_2 = 2\alpha = 0.828427125\ldots$, $p_3 = 3\alpha - 1 = 0.242640687\ldots$, $p_4 = 4\alpha - 1 = 0.656854250\ldots$. The numbers p_k are all different (why?). Now place in A_1 all those p_k which are less than $1 - \alpha = 0.585786438\ldots$, in A_2 all these p_k which are between $1 - \alpha$ and 1 (p_k can never be equal to $1 - \alpha$ or 1. Why?) For instance p_0, p_1 and p_3 are in A_1, p_2 and p_4 are in A_2. Finally put all the rest of the numbers between 0 and 1 (including 1) into A_3. Clearly $A_1 \cup A_2 \cup A_3 = [0, 1]$ is a partition of [0,1] into three pairwise disjoint subsets.

To obtain the corresponding decomposition of (0,1] define B_1 to be the set of all those p_k which are between α and 1, that is $\alpha \leq p_k < 1$, B_2 to be the set of all those p_k which are strictly between 0 and α, that is $0 < p_k < \alpha$, and B_3 to be the same as A_3. For instance p_1, p_2 and p_4 are in B_1, p_3 is in B_2, and $p_0 = 0$ is in neither of the sets B_1, B_2, B_3, so that $B_1 \cup B_2 \cup B_3 = (0, 1]$ is a partition of (0,1]. All we have to show now is that B_1 is a translation of A_1 and B_2 is a translation of A_2.

Suppose $p_k \in A_1$, that is $0 \leq p_k < 1 - \alpha$, then adding α to all members of the inequality, $\alpha \leq \alpha + p_k < 1$. Therefore $k\alpha = [k\alpha] + p_k$, add $\alpha : (k+1)\alpha = [k\alpha] + p_k + \alpha$. As $p_k + \alpha < 1, p_{k+1} = p_k + \alpha$, $p_{k+1} \in B_1$. This shows that if we add α to any number in A_1 we get a number in B_1, so that the translation of A_1 by α is contained in B_1. We still have to show that we get all of B_1, that is no member of B_1 is missed out by this translation of A_1. The argument is very much the same as before, only we subtract now α from $p_k \in B_1$ and show that $p_k - \alpha$ is in A_1. Very similarly one can prove that A_2 is a translation of B_2 by $1 - \alpha$. The interested reader can fill in the details without much difficulty.

Going back to Tarski's circle-squaring problem: it has finally been settled last year by a Hungarian mathematician by the name of Miklos Laczkovich. His answer is: yes, the circular disc and the solid square do have congruent partitions, even though the number of necessary parts is quite formidable, about 10^{50} (1 followed by 50 zeros). In fact he proved a much stronger statement: not only that the shape of the second figure does not matter as long as its area is the same as the area of the disc, but more surprisingly, in the partitions of Laczkovich the pieces need not even be rotated but merely translated – a result that caught almost everyone who worked in the subject by surprise. In the original Banach-

Tarski proof for the (3-dimensional) sphere rotation of the pieces was quite essential, and so were in the Bolyai type dissections.

I conclude with a remark which might interest some of our former competitors in the Mathematical Olympiad: when in 1982 the Australian team was sent to Budapest, Laczkovich was one of the moderators, and a very good one too. In his student years he himself was a keen and successful participant in various mathematical competitions. Perhaps competitions such as the IBM do improve your problem solving skills after all.

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"Newton could not admit that there was any difference between him and other men, except in the possession of such habits as ... perseverance and vigilance. When he was asked how he made his discoveries, he answered, "by always thinking about them;" and at another time he declared that if he had done anything, it was due to nothing but industry and patient thought: "I keep the subject of my inquiry constantly before me, and wait till the first dawning opens gradually, by little and little, into a full and clear light."

W. Whewell in **History of the Inductive Sciences**, Book 7, Chapt.2.

"His secretary records that his forgetfulness of his dinner was an excellent thing for his old housekeeper, who "sometimes found both dinner and supper scarcely tasted of, which the old woman has very pleasantly and mumpingly gone away with." On getting out of bed in the morning, he has been discovered to sit on his bedside for hours without dressing himself, utterly absorbed in thought."

James Parton, **Sir Isaac Newton.**