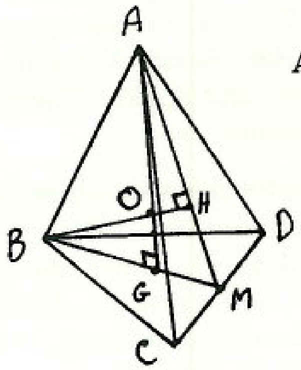
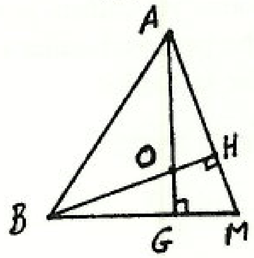
## SOLUTIONS OF PROBLEMS 793 - 804

**Q.793** The vertices of a regular tetrahedron lie on a sphere of radius $R$, and its faces are tangential to a sphere of radius, $r$. Calculate $R/r$.




**ANSWER** If $G$ is the foot of the perpendicular from $A$ to the face $BCD$, then $G$ is the centroid of the equilateral triangle $BCD$. (This is obvious from the symmetry of the figure, or by observing that the right angled triangles $\triangle ABG$, $\triangle ACG$, and $\triangle ADG$ have equal hypotenuses, and common side $AG$, so they are congruent, whence $BG = CG = DG$; etc).




Thus $G$ divides the line interval joining $B$ and $M$, the mid-point of $CD$, in the ratio 2:1. Similarly the foot of the perpendicular from $B$ to the face $ACD$ is $H$, dividing $AM$ in the ratio 2:1.

Since the lines $BH$ and $AG$ both lie in the plane of $\triangle ABM$, they intersect at the orthocentre, $O$, of this triangle. It is easy to see that $OA = OB$ and $OH = OG$ in this triangle (Fig.2).

[For example, $AM = BM$, (medians of congruent equilateral triangles), so the third altitude $OM$ bisects both the base $AB$ of this isosecles triangle, and the angle $M$. Using the symmetry about this line we can prove both assertions.] Note that from similar triangles $\triangle OGB$ and $\triangle MHB$

$$\frac{OG}{OB} = \frac{MH}{MB} = \frac{MG}{MB} = \frac{1}{3} = \frac{OG}{OA}.$$

Similarly one sees that the point $O$, dividing $AG$ in the ratio 3:1, is equidistant from $A$, $B$, $C$ and $D$, and also equidistant from all the faces of the tetrahedron. It is therefore the centre of both spheres referred to in the question,

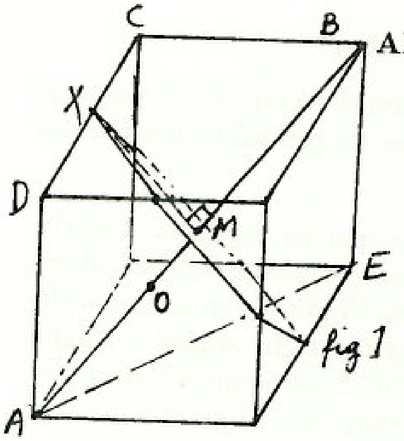
$$\text{and } \frac{R}{r} = \frac{OB}{OG} = \frac{3}{1}.$$

**Correct Solution:** A. Henderson (Sydney Grammar School)

**Q.794** $A$ and $B$ are opposite vertices of a cube of side length 1 unit.

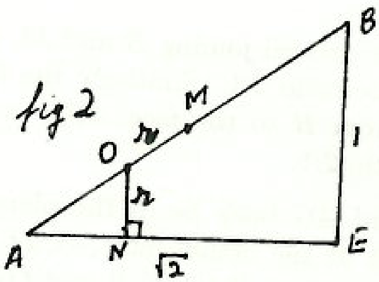
(i) Prove that the mid points of the six edges not containing either $A$ or $B$ all lie on a plane.

(ii) The cube is cut into two pieces along this plane. Find the radius of a sphere which touches that plane and three faces of the cube.
(i.e. of the largest sphere which fits inside either of the two pieces.)




**ANSWER**

(i) Let $X$ be one of the six points, the mid point of side $CD$ (see fig.1). Then $AX^2 = AD^2 + DX^2 = BC^2 + CX^2 = BX^2$. Thus $X$ is equidistant from $A$ and $B$, so the join of $X$ to $M$, the mid point of $AB$, is perpendicular to $AB$. Hence $X$ lies in the plane which bisects $AB$ at right angles. The same is true for all of the six points.




(ii) The centre, $O$, of a sphere touching the three faces meeting at $A$, being equidistant from these faces lies on the line joining $A$ to $B$. Since the perpendicular distance, $OM$, is the shortest distance from $O$ to the plane of the six points in (i), if the sphere touches this plane also its radius is equal to $OM$. In fig 2., $ON$ is the perpendicular to the base of the cube. By similar triangles

$\frac{1}{\sqrt{3}} = \frac{BE}{AB} = \frac{ON}{OA} = \frac{ON}{AM-OM} = \frac{r}{\frac{\sqrt{3}}{2}-r}$.

Solving for $r$, $r = \frac{\sqrt{3}}{2(\sqrt{3}+1)} = \frac{3-\sqrt{3}}{4}$.

[Comment: This is the radius of the largest sphere that lies inside one half of the cube. A sphere which touches the three faces meeting at $D$, say, as well as the six point plane turns out to have radius $\frac{\sqrt{3}-1}{4}$].

**Q.795** If $a, b, c$ are positive numbers such that $a^2 + b^2 = c^2$ prove that

$$\log_{c+b} a + \log_{c-b} a = 2 \log_{c+b} a \log_{c-b} a$$

**ANSWER** Let $\log_{c+b} a = v$ and $\log_{c-b} a = v$. Then $(c+b)^v = a$, whence $c + b = a^{\frac{1}{u}}$. Similarly $c - b = a^{\frac{1}{v}}$

$$a^2 = c^2 - b^2 = (c+b)(c-b) = a^{\frac{1}{u}} a^{\frac{1}{v}} = a^{\frac{1}{u}+\frac{1}{v}}$$

From this $\frac{1}{u} + \frac{1}{v} = 2$, giving $u + v = 2uv$, as required.

**Correct solution:** A. Henderson (Sydney Grammar School)

**Q.796** Find positive numbers $x, y$ such that

$$\begin{cases} x^{x+y} & = y^{x-y} \\ x^2 y & = 1 \end{cases}$$

**ANSWER**

$$x^{x+y} = y^{x-y} = \left(\frac{1}{x^2}\right)^{x-y} = \frac{1}{x^{2x-2y}}.$$

$$x^{3x-y} = 1.$$

EITHER $x = 1$ OR $3x - y = 0$.
If $x = 1$, then $y = \frac{1}{x^2} = 1$.
If $3x - y = 0$, $3x = \frac{1}{x^2}$ whence $x^3 = \frac{1}{3}$, $x = \frac{1}{\sqrt[3]{3}}$, $y = \sqrt[3]{9}$.

Thus there are two solutions:- $(x,\ y) = (1,\ 1)$, or $(\frac{1}{\sqrt[3]{3}},\ \sqrt[3]{9})$.

**Correct solution:** A. Henderson (Sydney Grammar School)

**Q.797** Let $a_1, a_2, \cdots, a_n$ be any list of non-zero numbers such that for every $k \geq 3$

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \cdots + \frac{1}{a_{k-1}a_k} = \frac{k-1}{a_1a_k}.$$

Prove that the list is an arithmetic progression.

**ANSWER** Let $a_2 = a_1 + d$.
Taking $k = 3$,

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} = \frac{2}{a_1a_3}, \text{whence (multiplying by } a_1a_2a_3)$$
$$a_3 + a_1 = 2a_2$$
$$a_3 = a_1 + 2(a_2 - a_1) = a_1 + 2d.$$

Now let us assume that, for some natural number $m$,

$$a_m = a_1 + (m-1)d. \qquad (1)$$

(We already have shown that this is true for $m = 3$). We will show that it is then true that $a_{m+1} = a_1 + md$.

Taking $k = m$ and $k = m + 1$ in the given equation and subtracting gives

$$\frac{1}{a_ma_{m+1}} = \frac{m}{a_1a_{m+1}} - \frac{m-1}{a_1a_m}$$
$$\frac{m-1}{a_1a_m} = \frac{1}{a_{m+1}}\left(\frac{m}{a_1} - \frac{1}{a_m}\right) = \frac{ma_m - a_1}{a_{m+1}a_1a_m}$$
$$a_{m+1} = \frac{(ma_m - a_1)}{m-1} = \frac{m(a_1 + (m-1)d) - a_1}{(m-1)}$$
$$= a_1 + md.$$

Since the assertion (1) is true when $m = 2$, and, as we have just shown, it remains true when $m$ is replaced by $m+1$, it is true for all natural numbers $m \geq 2$. Hence the list is an arithmetic progression.

**Correct solution:** A. Henderson (Sydney Grammar School)

**Q.798** Prove that for every positive integer $m$,

$$\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\cdots+\frac{1}{3m+1}>1$$

**ANSWER** This also can be proved by mathematical induction. When $m=1$ the result is true, since $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}>1$. Let $S_k=\frac{1}{k+1}+\frac{1}{k+2}+--+\frac{1}{3k+1}$ for all $k\geq 1$.

$$\begin{aligned}S_{k+1}-S_k&=\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}-\frac{1}{k+1}=\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}-\frac{3}{3k+3}\\&=\left(\frac{1}{3k+2}-\frac{1}{3k+3}\right)+0+\left(\frac{1}{3k+3}-\frac{1}{3k+4}\right)\\&=\frac{1}{(3k+2)(3k+3)}-\frac{1}{(3k+3)(3k+4)}>0.\end{aligned}$$

$\therefore S_{k+1}>S_k$ for $k=1,\ 2,\ldots$
Since $S_1>1$, it follows that $S_m>1$ for all $m\in\mathbb{N}$.

**An alternative solution** has been submitted by A. Henderson (Sydney Grammar School) who writes:

Consider the $2m+1$ distinct positive numbers: $m+1, m+2, m+3,\cdots,3m+1$. By the arithmetic/harmonic mean (strict) inequality,

$$A.M. > H.M.$$

$$2m+1>\frac{2m+1}{\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{3m+1}}$$

$$1>\frac{1}{\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{3m+1}}$$

$$\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{3m+1}>1$$

**Q.799** Two identical containers, $A$ and $B$, each of capacity $L$ litres, together contain a total of $L$ litres of alcohol. First $A$ is filled to the top with water, and the contents stirred thoroughly. Then liquid is poured from $A$ into $B$ until it is full. Finally $(2/5)L$ litres of the new mixture in $B$ is transferred to $A$ without spillage. If $A$ now contains $\frac{L}{15}$ litres more alcohol than $B$, find how many litres of alcohol were in each container originally.

**ANSWER** Let $A$ initially contain $x$ litres of alcohol, and $B$ contain $L-x$ litres. After $A$ is topped up with water, $x$ litres of the mixture transferred to $B$ contain $\frac{x}{L}\times x$ litres of alcohol.
Hence at this stage $A$ contains $x-\frac{x^2}{L}$ litres of alcohol, and $B$ contains $(L-x+\frac{x^2}{L})$ litres of alcohol. Since $\frac{2}{5}$ $L$ litres can be transferred to $A$ without spillage at this stage, $x\geq\frac{2}{5}L$.

Since $\frac{2}{5}(L - x + \frac{x^2}{L})$ litres of alcohol is transferred in this operation, the amount of alcohol in $A$ and $B$ respectively is finally

$$\frac{2}{5}(L - x + \frac{x^2}{L}) + x - \frac{x^2}{L} \text{ and } \frac{3}{5}(L - x + \frac{x^2}{L})$$

$$\text{Solving } \frac{2}{5}(L - x + \frac{x^2}{L}) + x - \frac{x^2}{L} = \frac{3}{5}(L - x + \frac{x^2}{L}) + \frac{L}{15}$$

$$\text{yields } 9x^2 - 9xL + 2L^2 = 0$$

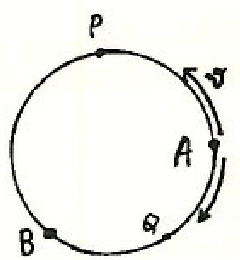
$$\frac{x}{L} = \frac{9 \pm \sqrt{9^2 - 72}}{18} = \frac{1}{3} \text{ or } \frac{2}{3}.$$

Since $x \geq \frac{2}{5}L$, the only valid solution is $x = \frac{2}{3}L$.
Initially $A$ held $\frac{2}{3}L$ litres of alcohol, and $B$ held $\frac{1}{3}L$ litres.

**Correct solution:** A. Henderson (Sydney Grammar School)

**Q.800** Two bodies move in opposite directions AROUND A CIRCULAR TRACK, one at constant speed $v$ m/sec. The speed of the other increases at a constant rate $a$ m/sec/sec. At time $t = 0$, the bodies are of the same point $A$ and the second one is momentarily at rest. In how many seconds does their first meeting occur, if their second meeting is again at the point $A$?

**ANSWER** (I apologise that gremlins gobbled up the words "AROUND A CIRCULAR TRACK" which have now been reinserted in this question. Only clairvoyants could have made any sense of the problem as it appeared.)




Let $d$ be the distance (in metres) round the track, and suppose their first meeting, at $B$, occurred after $t_1$ secs. Then

$$d = \text{ length } APB + \text{ length } AQB$$
$$= vt_1 + \frac{1}{2}at_1^2 \qquad (1)$$

Since their second meeting, after $t_2$ secs, occurred at $A$,

$$d = vt_2 \qquad (2)$$
$$\text{and} \quad d = \frac{1}{2}at_2^2 \qquad (3)$$

From (2) and (3) $d = \frac{1}{2}a\left(\frac{d}{v}\right)^2$, yielding $d = \frac{2v^2}{a}$.
Substitution in (1) gives $\frac{1}{2}at_1^2 + vt_1 - \frac{2v^2}{a} = 0$ from which $t_1 = \frac{v}{a}(-1 + \sqrt{5})$, since the negative root must be rejected.

Hence the first meeting occurred after $\frac{v}{a}(\sqrt{5} - 1)$ seconds.

**Q.801** Show that it is impossible to find three whole numbers $u, v, w$ such that

$$u^2 + v^2 = 3w^2$$

**ANSWER** A. Henderson (Sydney Grammar School, year 9) writes:

Any square is congruent mod 4 to either 0 or 1. Since we may cancel any common factors of $u, v$, and $w$, we may assume that $u^2, v^2$, and $w^2$ are not all congruent to 0. This clearly leaves no solutions.

**Q.802** How many different triangles (i.e. no two congruent) have perimeter 150cms, and every side a whole number of centimetres in length?

**ANSWER** Let the side lengths in cms be $a, b, c$ with $a \leq b \leq c$. Then $3a \leq a + b + c$, whence $a \leq 50$.

Denote by $N_a$ the number of the triangles in the set with smallest side equal to $a$. (e.g. $N_{50} = 1$ since clearly the equilateral triangle is the only such triangle with shortest side equal to 50cms). For each $a$ in $\{1, 2, ..., 50\}$ we must have $a \leq b$, $a + b \geq 76$ (two sides of a triangle must exceed the third) and $b \leq \frac{150-a}{2}$. For $1 \leq a \leq 37$, $\max\{a,\ 76 - a\} = 76 - a$ and $N_a$ is the number of whole numbers $b$ such that $76 - a \leq b \leq \frac{150-a}{2}$.

When $a$ is even both end points are included and

$$N_a = 1 + \frac{150 - a}{2} - (76 - a)$$
$$= \frac{a}{2}.$$

When $a$ is odd, $\frac{150-a}{2}$ is not an integer, and $N_a = 1 + \frac{149-a}{2} - (76 - a) = \frac{a-1}{2}$. For $38 \leq a \leq 50$, max $\{a,\ 76 - a\} = a$, and $N_a$ is the number of values of $b$ such that $a \leq b \leq \frac{150-a}{2}$.

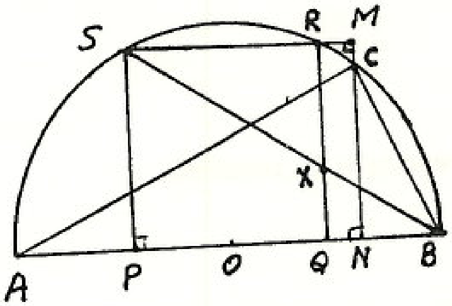
When $a$ is even $N_a = 1 + \dfrac{150 - a}{2} - a = 76 - \dfrac{3a}{2}$.

When $a$ is odd $N_a = 1 + \dfrac{149 - a}{2} - a = 76 - \dfrac{3a + 1}{2}$.

Thus the total number of triangles

$$= [N_1 + N_2 + N_3 + N_4 + ... + N_{36} + N_{37}]$$
$$+ [N_{38} + N_{39} + N_{40} + N_{41} + N_{42} + ... + N_{48} + M_{49} + N_{50}]$$
$$= [0 + 1 + 1 + 2 + ... + 18 + 18] + [19 + 17 + 16 + 14 + 13 + ... + 4 + 2 + 1]$$
$$= [2\sum_{k=1}^{18} k] + [\sum_{k=1}^{19} k - \sum_{k=1}^{6} 3k.]$$
$$= 2 \times \frac{18 \times 19}{2} + \frac{19 \times 20}{2} - 3\frac{6 \times 7}{2}$$
$$= 342 + 190 - 63. = 469.$$

**Q.803** In the figure $ABCRSA$ is a semicircle and $PQRS$ is a square.
Area $\triangle ABC$ = Area $PQRS$.
Prove that $X$, the point of intersection of $BS$ and $QR$, is the incentre of $\triangle ABC$.




**ANSWER** We take the unit of length equal to the radius of the circle. Let $O$ be the centre of the circle, $CN \perp AB$, and $CM \perp SR$ produced. Let $OP = OQ = x$, whence $PS = 2x$. Since $OP^2 + PS^2 = OS^2 = 1$, $x^2 + 4x^2 = 1$, so $x = \frac{1}{\sqrt{5}}$. The area of the square, $PS^2$, is equal to $(2x)^2 = \frac{4}{5}$ = area $\triangle ABC = \frac{1}{2}CN \times AB = CN$. Since $ON^2 + CN^2 = OC^2 = 1$ we calculate $ON = \frac{3}{5}$.
We prove first that $SB$ bisects $A\hat{B}C$ by showing that the chords $AS$ and $CS$ are equal in length:-

$$AS^2 = AP^2 + PS^2 = (OA - OP)^2 + PS^2 = (1 - \frac{1}{\sqrt{5}})^2 + (\frac{2}{\sqrt{5}})^2 = 2 - \frac{2}{\sqrt{5}}$$

$$CS^2 = SM^2 + CM^2 = (OP + ON)^2 + (SP - CN)^2 = (\frac{3}{5} + \frac{1}{\sqrt{5}})^2 + (\frac{2}{\sqrt{5}} - \frac{4}{5})^2 = 2 - \frac{2}{\sqrt{5}}.$$

It remains to be proved that $XA$ also bisects $C\hat{A}B$.

$$\tan C\hat{A}B = \frac{CN}{AN} = \frac{4}{5} / \left(1 + \frac{3}{5}\right) = \frac{1}{2}.$$

Let $t = \tan \frac{1}{2} C\hat{A}B$. Then $\frac{1}{2} = \tan C\hat{A}B = \frac{2t}{1-t^2}$. From this we obtain
$t^2 + 4t - 1 = O$, and $t = -2 + \sqrt{5}$ (1)

Now $\frac{QX}{PS} = \frac{BQ}{BP}$ yields $QX = \frac{1-\frac{1}{\sqrt{5}}}{1+\frac{1}{\sqrt{5}}} \times \frac{2}{\sqrt{5}}$; whence

$$\tan X\hat{A}Q = \frac{XQ}{AQ} = \frac{1-\frac{1}{\sqrt{5}}}{1+\frac{1}{\sqrt{5}}} \times \frac{2}{\sqrt{5}} \times \frac{1}{1+\frac{1}{\sqrt{5}}} = -2 + \sqrt{5}$$

Comparing with (1), we see that $X\hat{A}Q$ is indeed equal to $\frac{1}{2}C\hat{A}B$, completing the verification that $X$ is the intersection of the angle bisectors of $\triangle ABC$, as required.

**Q.804** $a_0, a_1, a_2, a_3, \cdots, a_n, \cdots$ is an increasing sequence of real numbers ($a_n < a_{n+1}$ for all $n$) defined by $a_{n+1} = 2^n - 3a_n$, $n = 0, 1, 2 \cdots$.
Find all possible values of $a_0$.

**ANSWER**

$$a_1 = 2^0 - 3a_0 = 1 - 3a_0$$
$$a_2 = 2^1 - 3a_1 = 2 - 3 + 3^2 a_0$$
$$a_3 = 2^2 - 3a_2 = 2^2 - 3 \times 2^1 + 3^2 - 3^2 a_0$$
$$a_4 = 2^3 - 3a_3 = 2^3 - 3 \times 2^2 + 3^2 \times 2^1 - 3^3 + 3^4 a_0.$$

Recognizing the pattern we deduce

$$\begin{aligned} a_n &= (2^{n-1} - 3 \times 2^{n-2} + 3^2 \times 2^{n-3} ... + (-1)^{n-1}3^{n-1}) + (-1)^n 3^n a_0 \\ &= 2^{n-1}\left[\frac{1-(\frac{-3}{2})^n}{1-(\frac{-3}{2})}\right] + (-1)^n 3^n a_0 \quad \text{(formula for summing a G.P)} \\ &= \frac{2^n}{5} - (-1)^n\frac{3^n}{5} + (-1)^n 3^n a_0 \\ &= \frac{2^n}{5} + (-1)^n(a_0 - \frac{1}{5})3^n. \end{aligned}$$

If $a_0 = \frac{1}{5}$, this gives $a_n = \frac{2^n}{5}$ which is an increasing sequence, as required.
If $A = a_0 - \frac{1}{5} \neq 0$

$a_n = (-1)^n 3^n A \left(1 + \frac{(-1)^n}{5A}(\frac{2}{3})^n\right)$.

By taking $N$ sufficiently large one can ensure that $\frac{(\frac{2}{3})^n}{5|A|} < 1$ for all $n \geq N$, so that $\left(1 + \frac{(-1)^n}{5A}(\frac{2}{3})^n\right) > 0$.
Hence for $n > N$ the terms $a_n$ have the same sign as $(-1)^n 3^n A$; i.e. they are alternately positive and negative. The sequence $\{a_n\}$ cannot be increasing unless $A = 0$; i.e. the only value of $a_0$ for which $\{a_n\}$ is increasing is $a_0 = \frac{1}{5}$.

**Correct solution:** A. Henderson (Sydney Grammar School)