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CATASTROPHES

By Michael Cowling*

The subject of this paper is the mathematical theory of catastrophes.

A catastrophe is a disaster. But there is, implicit in the word, the idea of a sudden change for the worse. If Wall Street share values drop by 30% in one year, economists talk of a recession, while if they drop 30% in one day, they talk of a catastrophe. The tectonic plates on the surface of the earth move continually, and along fault lines motion of several meters in one century are not unusual. But if these hard plates shift by ten centimeters in a few minutes, then we have a major earthquake on our hands. Again, it is the abruptness of the change that provokes the catastrophe. Similarly, stressed metals bend or snap as they are deformed slowly or sharply.

The mathematical theory of catastrophes aims to explain mathematically how a continuously varying system can exhibit sudden changes in character.

Here is a first simple example. Imagine we have a system involving a bead sliding freely along a moving wire. Suppose that the wire lies in a two-dimensional plane, and at time t, the shape of the wire is given by the equation

$$y = x^3 + 3tx,$$

where y is a vertical direction and x a horizontal direction. At a fixed t, thinking of y as a function of x, we have

$$y'(x) = 3x^2 + 3t.$$

The graph $y = x^3 + 3tx$ has "critical points" where $y'(x) = 0$. In order to sketch $y = x^3 + 3tx$, we should distinguish three cases:

$t < 0$	2 critical points	$x = \pm\sqrt{-t}$
$t = 0$	1 critical point	$x = 0$
$t > 0$	0 critical points	

The graphs for these three cases are sketched in Figure 1.

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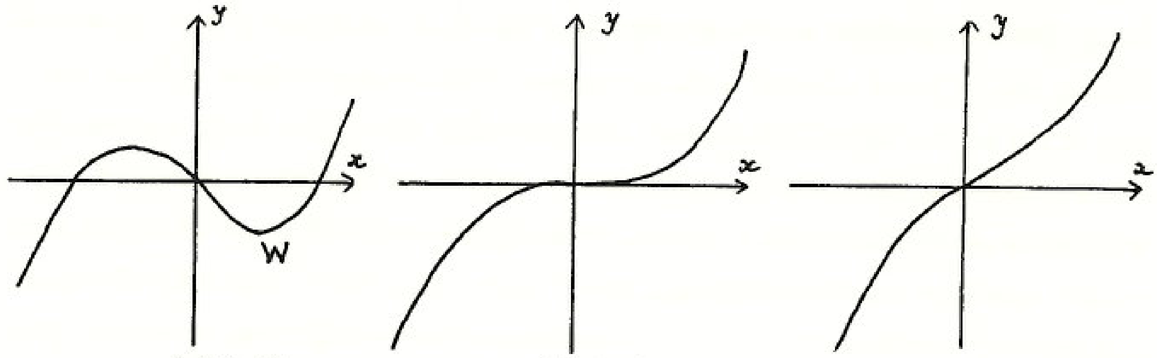


(a) $t < 0$ (b) $t = 0$ (c): $t > 0$

Figure 1: the graph of $y = x^3 + 3tx$

Suppose that the bead starts out at $(1, -2)$ when $t = -1$. As time evolves, the bead stays at the bottom of the "potential well", and as long as $t < 0$, the bead is in "stable equilibrium". This means that, if we shift it a little to one side or the other, the force of gravity tends to pull the bead back towards the point $W = ((-t)^{1/2},\ 2(-t)^{3/2})$, the bottom of the well. However, when $t = 0$, the "well" disappears, and the bead, now at $(0, 0)$, is in "unstable equilibrium". This means that a slight shift in the position of the bead will send it careering along the wire, though if it is at rest at $(0, 0)$, it will stay there. Finally, when $t > 0$, there is no "equilibrium point" and the bead disappears down the wire.

Now let us look at a second example, again with a bead on a wire, this time described by the equation

$$y = x^4 + 2tx^2 + x.$$

For this equation

$$y' = 4x^3 + 4tx + 1,$$

and there are critical points when

$$4x^3 + 4tx + 1 = 0.$$

Even though we may be unable to solve this equation for x, we can decide how many critical points there are. Let us digress a moment to do this. Before digressing, we remark that this last equation is a "cubic" equation, and there is a general formula for solving these. But for equations involving powers of x like x^5, x^6, or higher powers, no general formula can be found. The proof that no general formula exists is quite difficult, and is part of the subject called Galois theory. You may think that such a proof is useless, but it has meant that thousands of man hours have been saved as mathematicians no longer work on the problem of finding a solution. (Actually, there are still people who try to solve quintic equations, or, more commonly, trisect angles using only a ruler and a compass, and university mathematics departments sometimes have to help them by finding errors in their work, so one could wish that Galois theory were easier and more widely known).

Now we digress. Let

$$z(x) = 4x^3 + 4tx + 1.$$

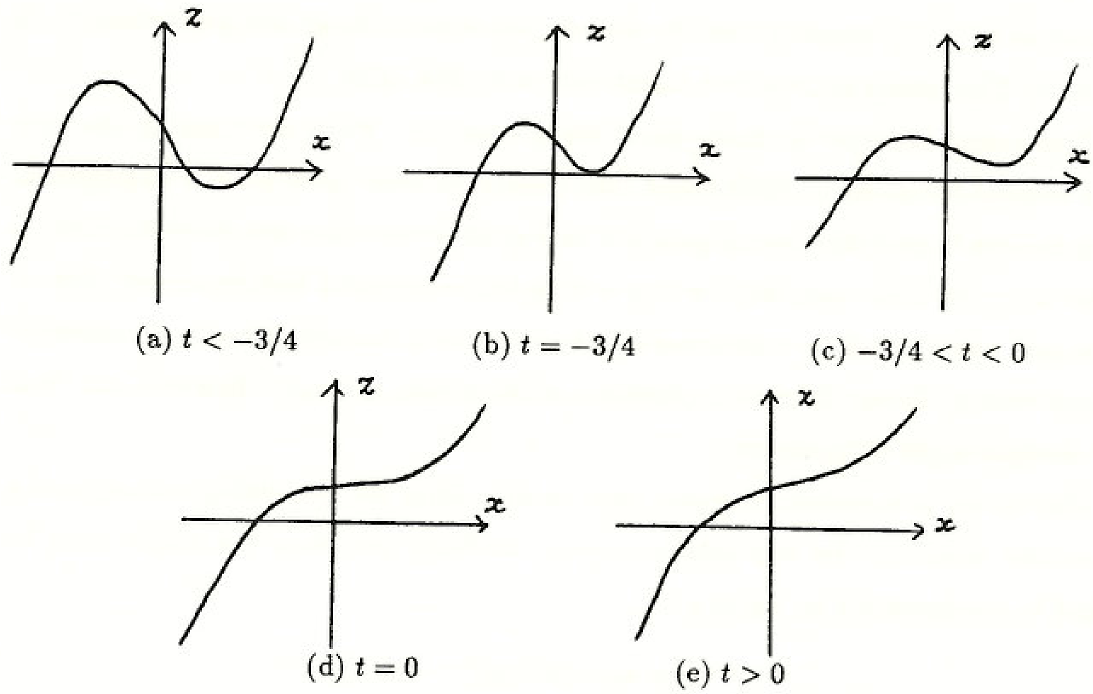
To find out how many solutions $z(x) = 0$ has, we ask how often $z'(x)$ changes sign. For if $z'(x)$ is always positive, then $z(x)$ is an increasing function, and it can have at most one root. Observe that

$$z'(x) = 12x^2 + 4t$$

which has no zeroes if $t > 0$, one zero, $x = 0$, if $t = 0$, and two zeroes, $x = \pm(-t/3)^{1/2}$, if $t < 0$. The critical values of $z(x)$, i.e., the values of $z(x)$ at critical points, are

$$\begin{aligned} & 4(\pm(-t/3)^{1/2})^3 + 4t(\pm(-t/3)^{1/2}) + 1 \\ &= \pm 4(-t/3)^{1/3}[-t/3 + t] + 1 \\ &= 1 \mp (-4t/3)^{3/2}. \end{aligned}$$

Notice that if $0 < -4t/3 < 1$, then both these critical values are positive; if $-4t/3 = 1$, then one critical value is 0, and if $-4t/3 > 1$, then one critical value is negative and the other positive. The following diagrams (Figure 2) sketch the graph of z for different values of t.

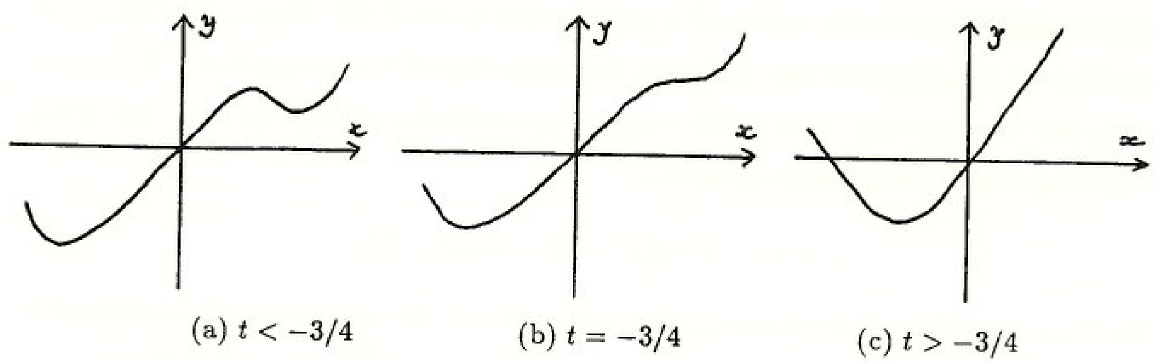


(a) $t < -3/4$ (b) $t = -3/4$ (c) $-3/4 < t < 0$

(d) $t = 0$ (e) $t > 0$

Figure 2: the graph of $Z = 4x^3 + 4tx + 1$

Now we return to the function y. Since we know how many times $y'(x) = 0$, we can draw a rough sketch of the graph of y, as in Figure 3.



(a) $t < -3/4$ (b) $t = -3/4$ (c) $t > -3/4$

Figure 3: the graph of $y = x^4 + 2tx^2 + x$

Again there is a potential well (to the right of y axis) that disappears when t reaches the value $-3/4$, and a bead on this wire would exhibit similar "catastrophic" behaviour to that we described in the first example. When the well disappears, there is an inflexion point, marked I on Figure 3(b).

Mathematicians describe such examples as "catastrophes". The first example is a catastrophe at $(0,0)$, meaning that the abrupt behavioural change occurred when $t = 0$ and $x = 0$. The second example is a catastrophe at $(-3/4, 1/2)$.

Two comments must be made about these examples. First, the bead on the wire was a convenient model to think about, but there have been more general applications. Sharp changes in economic, sociological and biological systems have also been described in similar ways. In these examples, the axes will no longer represent horizontal and vertical, but supply and demand, or concentrations of substances in biological media, or whatever. I do not wish to discuss different applications of these ideas at length. However, one "real life" example might be pertinent.

One model of economics suggests that, in discussing the availability and price of a commodity, whether it be disposable tissues or van Gogh paintings, the supply s can be related to the demand d by the rule

$$s = s_0 + (d - d_0)^3.$$

In this formula, s_0 and d_0 are "steady state" supply and demand. The justification is that when demand increases, sellers first maintain supplies and increase their profits (so supply increases less than demand) but as these profits increase, it becomes better to increase supply substantially and cash in on the extra demand. Of course, this equation is simplistic for many reasons; one of these is that consumption is not considered. If demand is over the "steady state" level, more is being consumed and, as time goes on, this tends to decrease supply. So a better equation is

$$s = s_0 + (d - d_0)^3 - C(t - t_0)(d - d_0),$$

where C is a positive constant and t_0 is some fixed time. We can rewrite this equation by putting

$$y = s - s_0 \qquad x = d - d_0 \qquad t_1 = C(t - t_0),$$

to get the equation

$$y = x^3 - t_1 x.$$

This is like the equation of the first example, but with the sign of t_1 reversed. Now, before the critical time t_0, supply and demand are related by a graph like Figure 1(c), while after time t_0, they are linked by a graph like Figure 1(a). In this figure, there are several values of x corresponding to the same values of y, i.e., several values of demand correspond to the same value of supply. It is suggested that economic catastrophes occur when "market confidence is shaken" and the demand drops to a lower value commensurate with the same supply, as in going from A to B in Figure 4. Note that the mathematical theory does not explain *why* market confidence is shaken. Note also that, even though this example and the earlier examples model rather different situations, from a purely mathematical point of view they are similar: at a particular value of t the character of a continuously varying graph changes. This is the mathematical essence of a catastrophe.

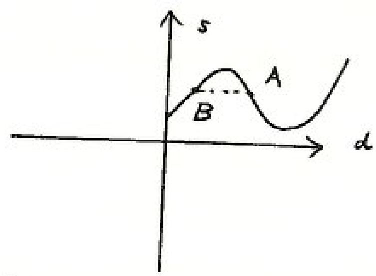
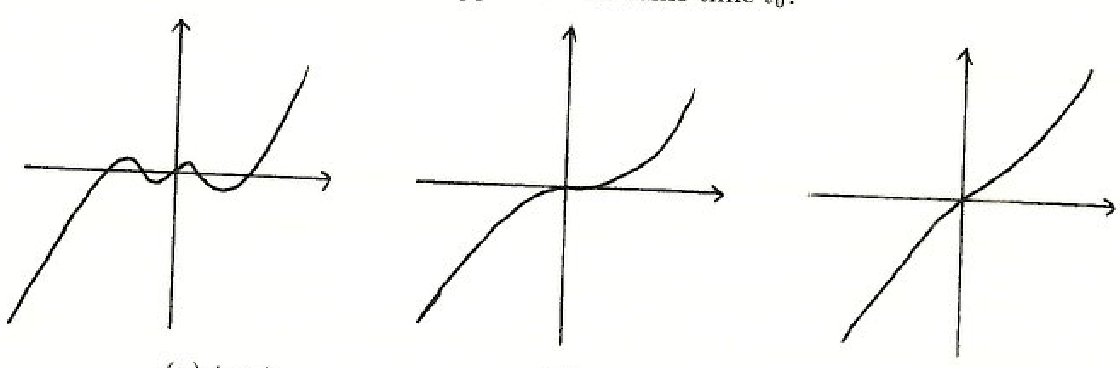


Figure 4: economic disaster

Now I would ask you to look again at examples 1 and 2, and to restrict your attention to the parts of the graph near to the potential well (actually, to the whole sketched graph in Figure 1, and to the parts of the graphs to the right of the y-axis in Figure 2). These look quite similar. In fact, in some sense, these "catastrophes" are "the same", in as much as a single "well" is disappearing. We can at least imagine a situation with graphs as in Figure 5 where several "wells" disappear at the same time t_0.



(a) $t < t_0$ (b) $t = t_0$ (c) $t > t.$

Figure 5: a different catastrophe

This would be an essentially different catastrophe. To clarify in what sense examples 1 and 2 involve the same catastrophe, note that there is a fixed inflection point when $x = y = 0$ in the first example, while in the second example there is an inflection point at $x = (-t/3)^{1/2}$, $y = [(-t/3)^2 + 2t(-t/3) + (-t/3)^{1/2}]$. The "catastrophes" happen when $t = 0$ in the first example and when $t = -3/4$ in the second. In the second example, we choose new coordinates

$$x_1 = x - (-t/3)^{1/2}$$
$$y_1 = y - [(-t/3)^2 + 2t(-t/3) + (t/3)^{1/2}]$$
$$t_1 = t/3 + 1/4$$

to move the inflection point to the origin, and the catastrophe time to 0. Substituting these in,

$$y = x^4 + 2tx^2 + x$$

becomes

$$y_1 = y_1^4 + x_1[1 - (1 - t_1/2)^{1/2}] + 2(1 - t_1/2)^{1/2}[x_1^3 + t_1 x].$$

Near to the point $x_1 = y_1 = t_1 = 0$, we may say that x_1^4 is negligeable relative to $(1 - t_1/2)^{1/2}x_1^3$, and discarding other "negligeable" terms, we obtain

$$y_1 \simeq 2x_1^3 + (9/4)t_1 x_1$$

($\simeq$ means approximately equal to). If we change the scale on the axes, by putting

$$x_2 = x_1,\ y_2 = y_1/2,\ t_2 = 9t_1/8,$$

we obtain the equation

$$y_2 \simeq x_2^3 + t_2 x_2,$$

which is the equation for the first example, with equality replaced by approximate equality.

In general, mathematicians identify catastrophes which can be turned into each other by such changes of variables and approximations.

A remarkable achievement of twentieth century mathematics was to show that "all possible" mathematical catastrophes could be classified, so that any mathematical catastrophes is as similar to one on the standard list as the second example is to the first. It is perhaps noteworthy that the list of catastrophes, discovered over the last few decades, also involves functions of many variables where human intuition is pushed to its limits. Last but not least, I would suggest, particularly for those of you who are intrigued by the new theory of chaos, that the mathematical theory of chaos is attempting to "classify chaos" in the same way that the mathematical theory of catastrophes classifies catastrophes.

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"I found at once, when I came to Cambridge, that a Fellowship implies 'original work', but it was a long time before I formed any definite idea of research. I had of course found at school, as every future mathematician does, that I would often do things much better than my teachers; and even at Cambridge I found, though naturally much less frequently, that I could sometimes do things better than the College lecturers. But I was really quite ignorant, even when I took the Tripos, of the subjects on which I have spent the rest of my life; and I still thought of mathematics as essentially a 'competitive' subject. My eyes were first opened by Professor Love, who taught me for a few terms and gave me my first serious conception of analysis. But the great debt which I owe to him – he was, after all, primarily an applied mathematician – was his advice to read Jordan's famous *Cours d'analyse*; and I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant. From that time onwards I was in my way a real mathematician, with sound mathematical ambitions and a genuine passion for mathematics."

from **A mathematician's apology**

by G.H. Hardy